THE ORDER OF APPROXIMATION TO FUNCTIONS
OF THE $Z_\alpha$ CLASS BY MEANS OF POSITIVE LINEAR OPERATORS

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Let $C_n(\varphi, \alpha)$ be the upper bound for deviations of periodic functions which form the Zygmund class $Z_\alpha$, $0 < \alpha < 2$ from a class of positive linear operators. A study is made of the conditions under which there exists a limit $\lim_{n \to \infty} n^\alpha C_n(\varphi, \alpha) = C(\varphi, \alpha)$. An explicit expression is given for the functions $C(\varphi, \alpha)$.

Let $\varphi(x)$ be a function given on the segment $[0, 1]$, and suppose

$$u_n(t) = \frac{1}{2A_n} \sum_{k=0}^{n} q\left(\frac{k}{n}\right) \varphi(t+k),$$

where

$$A_n = \sum_{k=0}^{n} q\left(\frac{k}{n}\right) \neq 0.$$

It is not difficult to see that

$$u_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \rho_{n,k} \cos kt \geq 0$$

and

$$\rho_{n,k} = \frac{A_{n,k}}{A_n}, \quad A_{n,k} = \sum_{s=0}^{n-k} q\left(\frac{s}{n}\right) q\left(\frac{s+k}{n}\right).$$

We put

$$L_n(f, x) = \frac{1}{n} \sum_{t=0}^{n-1} f(x+t) u_n(t) dt,$$

$$c_n(\varphi, \alpha) = \sup_{f \in Z_\alpha} \|f(x) - L_n(f, x)\|_{L_\alpha},$$

where $Z_\alpha(0 < \alpha \leq 2)$ is the class of $2\pi$-periodic functions $f(x)$ satisfying for all $x$ and $h$ the inequality

$$|f(x+h) + f(x-h) - 2f(x)| \leq 2|h|^{\alpha}.$$

For the operators (3) with the kernels (2) the following equations hold (see [1, 2])

$$c_n(\varphi, \alpha) = \frac{2}{\pi} \Gamma(\alpha) \sin \frac{\pi}{2} \sum_{k=0}^{n} \frac{\Delta \rho_{n,k}}{(k+1/2)^\alpha} + O\left(\sum_{k=0}^{n} \frac{|\Delta \rho_{n,k}|}{(k+1/2)^\beta}\right),$$

where $\rho_{n,0} = 1, \Delta \rho_{n,k} = \rho_{n,k} - \rho_{n,k-1}, \Delta \rho_{n,n} = \rho_{n,n}$, and (see [3]) $c_n(\varphi, 2) = 2(1-\rho_0, 0) + o(1-\rho_0, 0)$, if

$$\lim_{n \to \infty} \frac{1-\rho_{n,1}}{1-\rho_{n,0}} = 4.$$

For a narrower class of operators (3) with kernels (1) we prove below four theorems, from which follows the order of diminution of $c_n(\varphi, \alpha)$. These theorems generalize Theorems 5 and 6 of [2], in which $\varphi(x)$ is assumed to be continuously differentiable on $[0, 1]$. 

THEOREM 1. If the continuous function \( \varphi(x) \) has bounded variation and \( 0 < \alpha < 1 \), then

\[
\lim_{n \to \infty} n^\alpha c_n(\varphi, x) = c(\varphi, x),
\]

where

\[
c(\varphi, x) = \frac{2\Gamma(\alpha) \sin \frac{\alpha\pi}{2}}{\pi} \int_0^1 \int_0^{1-y} \frac{dy}{y^x} \int_0^y \varphi(x + y) dy \varphi(y). \tag{5}
\]

Proof. Since for \( 0 \leq k \leq n-1 \)

\[
\Delta \varphi_{n,k} = \frac{\Delta A_{n,k}}{A_n} = -\frac{1}{A_n} \sum_{k=0}^{n-k} \varphi\left(\frac{k}{n}\right) \Delta \varphi\left(\frac{k}{n} + \frac{1}{2n}\right),
\]

and \( \Delta \varphi_{n,n} = \frac{1}{A_n} \varphi(0) \varphi(1) \), it follows that

\[
n^2 \sum_{k=0}^{n-1} \varphi\left(\frac{k}{n} + \frac{1}{2n}\right) \varphi\left(\frac{k}{n}\right) = -\frac{1}{A_n} \sum_{k=0}^{n-k} \left(\frac{k}{n} + \frac{1}{2n}\right)^{-n} \times \sum_{k=0}^{n-k} \varphi\left(\frac{k}{n}\right) \varphi\left(\frac{k}{n} + \frac{1}{2n}\right) \varphi\left(\frac{k}{n}\right) + \frac{1}{A_n} \varphi(0) \varphi(1). \tag{6}
\]

We shall show that when \( n \to \infty \)

\[
\sum_{k=0}^{n-k} \varphi\left(\frac{k}{n} + \frac{1}{2n}\right) \varphi\left(\frac{k}{n}\right) \to \int_0^1 \int_0^{1-y} \frac{dy}{y^x} \int_0^y \varphi(x + y) dy \varphi(y). \tag{7}
\]

Let \( 0 < \delta < 1 \).

Then

\[
\left| \sum_{k=0}^{n-1} \left( \frac{k}{n} + \frac{1}{2n} \right)^{-n} - \int_0^1 \int_0^{1-y} \frac{dy}{y^x} \int_0^y \varphi(x + y) dy \varphi(y) \right|.
\tag{8}
\]

Since the sum in (8) is the integral Riemann-Stieltjes sum of the function \( y^{-\alpha} \varphi(x+y) \) with respect to the function \( y \varphi(x) \) for the triangular region bounded by the straight lines \( x = 0, \ y = 0, \text{and} \ x + y = 1 \), it is sufficient to prove that when \( n \to \infty \) and \( \delta \to 0 \)

\[
\sum_{k=0}^{n-1} \left( \frac{k}{n} + \frac{1}{2n} \right)^{-n} \int_0^1 \int_0^{1-y} \frac{dy}{y^x} \int_0^y \varphi(x + y) dy \varphi(y) \to 0.
\]

But

\[
\left| \sum_{k=0}^{n-1} \left( \frac{k}{n} + \frac{1}{2n} \right)^{-n} \int_0^1 \int_0^{1-y} \frac{dy}{y^x} \int_0^y \varphi(x + y) dy \varphi(y) \right| \leq MV \left[ \frac{\delta^{1-n}}{1-\alpha} + O(n^{-1}) \right] \to 0,
\]

where \( M = \max_x \varphi(x) \) and \( V \) is the total variation of the function \( \varphi(x) \) on \([0, 1]\).

Further,

\[
\left| \int_0^1 \int_0^{1-y} \varphi(x + y) dy \varphi(y) \right| \leq MV \int_0^1 \int_0^{1-y} \varphi(x + y) dy \varphi(y) \to 0.
\]

It can be proved in a similar way that

\[
\sum_{k=0}^{n-1} \varphi\left(\frac{k}{n} + \frac{1}{2n}\right)^{-n} \varphi\left(\frac{k}{n}\right) \to \int_0^1 \int_0^{1-y} \varphi(x + y) dy \varphi(y). \tag{9}
\]

Moreover,

\[
A_n \to \int_0^1 \varphi^3(x) dx. \tag{10}
\]