AN INVERSE PROBLEM FOR AN ORDINARY SECOND-ORDER DIFFERENTIAL EQUATION

Yu. E. Anikonov

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An inverse problem is solved for an ordinary second-order differential equation.

In this paper we consider an inverse problem for an ordinary second-order differential equation which is as follows: it is required to find a function $\mu(y) < 0$ in $y \equiv 0$ if:

1. For any $k$, $0 \leq k \leq k_0$, the following problem has a unique solution $y(t, k)$:

$$\frac{dy}{dt} = k\mu(y), \quad 0 \leq t \leq 1, \quad y(0, k) = y(1, k) = 0.$$

2. $\frac{dy}{dt} \big|_{t=0} = \varphi(k), \quad 0 \leq k \leq k_0$ and the function $\varphi(k)$ is specified.

Consider first problems which can be reduced to this form.

a) Let $D$ be a region of a Euclidean space $E^n$ of variables $y = (y_1, y_2, \ldots, y_n)$, $n = 2, 3, \ldots$, $\Gamma$ its boundary. It is of geophysical interest to determine the Riemannian metric $ds^2 = \lambda^2(y) |dy|^2$, in $D$ if the boundary conditions for the equations of the geodesics of this metric are specified at points of $\Gamma$.

Let $\gamma$ be a geodesic of the metric $ds^2 = \lambda^2(y) |dy|^2$. We introduce the parameter $\xi$ on $\gamma$, putting $\xi = \int \frac{|dy|}{\lambda(y)}$. Then the Euler equations $\frac{\partial \lambda}{\partial y_i} |y'| = \frac{d}{dt} \left( \lambda \frac{y_i}{|y'|} \right)$ defining the geodesic $\gamma$ take the form

$$\frac{d^2y_i}{dt^2} = \frac{1}{2} \frac{\partial}{\partial y_i} \lambda^2(y), \quad i = 1, 2, \ldots, n. \quad (1)$$

If $\lambda(y) = \sum_{i=1}^{n} \lambda_i^2(y_i)$, Eqs. (1) can be separated:

$$\frac{\partial y_i}{\partial x^2} = \frac{1}{2} \frac{\partial}{\partial y_i} \lambda_i^2(y_i), \quad i = 1, 2, \ldots, n. \quad (2)$$

Let $\frac{1}{2} \frac{\partial}{\partial y_i} \lambda_i^2(y_i)$ be denoted by $\nu_i(y_i)$ and assume that $\xi$ varies in the interval $0 \leq \xi \leq \sqrt{k}$, $0 \leq k \leq k_0$. Put $t = \frac{\xi}{\sqrt{k}}$. In this notation, Eqs. (2) can be rewritten as:

$$\frac{\partial y_i}{\partial x^2} = k\mu_i(y_i), \quad i = 1, 2, \ldots, n, \quad 0 \leq t \leq 1. \quad (3)$$

If now we add conditions 1 and 2 to the $i$-th equation of (3), we obtain the problem posed above.

b) Other inverse problems for partial differential equations can also be reduced to the form considered above; for example, one such is to find the function $\mu(u)$ in the equation

$$\Delta u = \sum_{i=1}^{n} \frac{\partial u}{\partial y_i^2} = k\mu(u)$$

if

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\[ u \mid_\Gamma = 0, \quad \frac{\partial u}{\partial n} \mid_\Gamma = \varphi (k), \quad |y| \leq 1, \]

where \( \Gamma \) is the sphere: \( |y| = 1 \), \( \partial u / \partial n \) is the derivative of \( u(y) \) along the normal to \( \Gamma \) and \( \varphi (k) \) is known.

Indeed, if we ignore derivatives of the first order \( \partial u / \partial y \), assuming them to be small, then \( \Delta u = k \mu (u) \) can be taken as Euler's equation for the functional

\[ I(u) = \int_{|y|<1} \left[ \sum_{n=1}^{N} \left( \frac{\partial u}{\partial y} \right)^2 + 2ka(u) \right] \frac{dy}{|y|^{n-1}}, \]

where \( \frac{d}{du} \mu (u) = \varphi (u) \).

On the other hand, assuming that \( u(y) \) depends only on \( r = |y| \), we can transform to a spherical co-ordinate system and obtain \( I(u) = c \int_0^1 \left[ u \cdot \mu (u) + 2ka(u) \right] dr \), where \( c \) is a constant. Here Euler's equation has the form: \( u_{rr} = k \mu (u) \) and we have \( u(1) = 0, \quad u_r(0) = 0, \quad -u(1) = \varphi (k) \). If we continue \( u(r) \) evenly in \( -1 \leq r \leq 0 \), the problem arises:

**THEOREM.** Let the function \( f(x) = 1 - \varphi^2(x^2) / x \) be continuously differentiable, positive, strictly monotonically decreasing and let \( f(0) = 1 \), while \( x(q) \) is defined by the equation \( f(x(q)) = q^2 \); then \( \mu (y) = (1/2) (d/du) \varphi^2 \), where \( \varphi (y) \) is the inverse of the function

\[ \varphi (y) = \sqrt{y} \int_0^1 \frac{1}{\sqrt{q^2 - y}}, \]

For any fixed \( p \), the equation \( \mu (y) = (1/2) (d/du) \varphi^2 \) has the first integral

\[ \frac{1}{\varphi^2 (y) - q}, \]

where \( q \) is a constant.

From (4) and (5) we have

\[ q = b^2 (z) - \left( \frac{dz}{d\zeta} \right)^2 = b^2 (0) - z^2 (0) = 1 - \frac{q^2 (z)}{x} = f(z). \]

Since by the hypothesis of the theorem \( f(x) > 0 \), \( q \) is positive. Put \( q = p^2 \). With this notation, the equation for \( z(\zeta) \) can be written as

\[ \frac{dz}{d\zeta} = \sqrt{b^2 (z) - p^2}, \]

where \( \frac{dz}{d\zeta} = \sqrt{b^2 (z) - p^2} \) is the equation for one half of the curve \( z = z(\zeta) \) (up to the peak) and \( \frac{dz}{d\zeta} = -\frac{1}{\sqrt{b^2 (z) - p^2}} \) is the equation for the other.

We see from (6) and (7) that for each \( p \leq \sqrt{f(y)} \), there is a curve \( z = z(\zeta) \).

Let \( x(p) \) be the greatest abscissa of \( z = z(\zeta) \) as a function of \( p \) and \( x(p) \) is greatest ordinate, i.e., \( z(x(p)) = x, \quad b(z(p)) = p \). The curve \( z = z(\zeta) \) is obviously symmetrical about the straight line \( \zeta = x(p)/2 \) for any fixed \( p \), and so from the equation we obtain, by integration,

\[ x(p) = 2 \int_{y_0}^{x(p)} \frac{dz}{\sqrt{b^2 (z) - p^2}}. \]