In this paper we give an example of two convex functions in $|\xi| > 1$ whose arithmetic mean is nonconvex. We calculate the radius of convexity of the sum of two convex functions; it is equal to $\sqrt{1 + \sqrt{2}}$. For functions $F(\xi) = \xi + b_1/\xi + \ldots$, where $F'(\xi) = f(\xi)/\xi$, if $f(\xi) = \xi + a_1/\xi + \ldots$ is univalent $|\xi| > 1$, then the radius of univalence is the root of the equation $4E(1/r)/K(1/r) + 1/r^2 = 3$.

1. On the Convexity of the Sum of Convex Functions

We consider the class $\Sigma$ of functions analytic in $1 < |\xi| < \infty$ having a simple pole at $\xi = \infty$ whose expansion in a neighborhood $\xi = 0$ has the form

$$F(\xi) = \xi + a/\xi + \ldots$$

In [1] it was proved that if $F_i(\xi) \in \Sigma$, $i = 1, \ldots, n$ are convex functions then the function

$$F(\xi) = \frac{\sum \alpha_i F_i(\xi)}{\sum \alpha_i},$$

where $\alpha_i, i = 1, \ldots, n$ are any positive numbers, is univalent. The question naturally arises: is $F(\xi)$ convex?

In this paper we present an example of a function

$$F(\xi) = \frac{1}{2} [F_1(\xi) + F_2(\xi)],$$

which is not convex although $F_1(\xi)$ and $F_2(\xi)$ are convex.

For $F_1(\xi)$ and $F_2(\xi)$ we choose the function with the required expansion in a neighborhood of $\xi = \infty$, knowing that

$$F_1'(\xi) = \exp(1/2 \xi^2),$$
$$F_2'(\xi) = \exp \left(-\int_0^{1/2} g(z) \frac{dz}{z^2} \right),$$

where $g(z)$ is some rational function in $|z| < 1$ with expansion in a neighborhood of $z = 0$

$$g(z) = b_2z^2 + b_3z^3 + \ldots,$$

*Definition of convexity: $f(\xi)$ in $\Sigma$ is called convex if it maps $|\xi| > 1$ onto a region with convex complement, that is, if we have the inequality

$$\text{Re } ((1 + \xi)f''(\xi)/f'(\xi)) > 0, \ |\xi| > 1.$$
satisfying the following conditions

\[ \Re g(z) \geq -1, \]  
\[ g(-1) = -1 + i\alpha, \]  
\[ x \cdot \Im \exp \left( \int_{y_0}^{x} \frac{g(z)}{z} \, dz \right) < 0. \]

It is not difficult to verify that in such a case \( F_i(z), i = 1, 2, \) are both convex functions and for

\[ F(z) = \frac{1}{2} [F_1(z) + F_2(z)] \]

we have the relation

\[ \Re \frac{F''}{F'} \bigg|_{x=-1} = -1 + \frac{\alpha^2}{1 + \exp \left( \frac{1}{2} + \int_{y_0}^{x} g(z)^{-1} \, dz \right)^2} \alpha \cdot \Im \exp \left( \int_{y_0}^{x} \frac{g(z)}{z} \, dz \right) < -1. \]

Hence, the above analysis will provide the desired example if there is a function \( g(z) = b_2z^2 + b_3z^3 + \ldots \) satisfying conditions (1a), (1b), and (1c).

We set

\[ g(z) = a_2z^2 + a_3z^3 + a_4z^4, \]
\[ a_k = \tilde{a}_k + i\delta_k, \quad k = 2, 3, 4, \]

We will satisfy condition (1b) by taking \( \tilde{a}_3 = a > 0, \tilde{a}_2 = a - 1, \tilde{a}_4 = 0. \)

In order to satisfy (1b) and (1c) we have to have

\[ \tilde{a}_2 - \tilde{a}_3 + \tilde{a}_4 = \alpha > 0, \tilde{a}_2/2 - \tilde{a}_3/3 + \tilde{a}_4/4 < 0. \]

It remains to guarantee condition (1a). In this case \( \Re g(e^{i\theta}) = \varphi(\theta) + \psi(\theta), \) where

\[ \varphi(\theta) = a \cdot \cos 3\theta + (a - 1) \cdot \cos 2\theta, \]
\[ \psi(\theta) = \tilde{a}_2 \cdot \sin 2\theta + \tilde{a}_3 \cdot \sin 3\theta + \tilde{a}_4 \cdot \sin 4\theta. \]

The absolute minimum of \( \varphi(\theta) \) on the interval \([0, 2\pi]\) occurs at \( \theta = \pi \). In this case \( \varphi(\pi) = -1, \) while \( \psi(\theta) \) has two local extrema on the interval \([0, 2\pi]\):

\[
A_{1,2} = a \left\{ \frac{2a \pm \sqrt{4a^2 + 36(1-a)^2}}{12(1-a)} \right\} + (a - 1) \left\{ \frac{2a \pm \sqrt{4a^2 + 36(1-a)^2}}{12(1-a)} \right\} - 3 \left[ 1 - \left( \frac{2a \pm \sqrt{4a^2 + 36(1-a)^2}}{12(1-a)} \right)^2 \right].
\]

Let \( A = \min(A_1, A_2) \). We now choose

\[ |\delta_2| + |\delta_3| + |\delta_4| < 1 - |A|. \tag{2} \]

Then the local minimum of the function \( \Re g(e^{i\theta}) \) is strictly larger than \(-1\). However it is necessary to impose one additional restriction on \( \psi(\theta) \) in order that in the neighborhood of \( \theta = \pi \) we have \( \Re g(e^{i\theta}) \geq -1 \): \( \psi'(\pi) = 0 \), that is,

\[ -2\tilde{a}_2 + 3\tilde{a}_3 - 4\tilde{a}_4 = 0. \]

Condition (1a) will be satisfied if \( a_1, i = 2, 3, 4, \) satisfy (2) and have the form

\[ \tilde{a}_2 = -9/M, \quad \tilde{a}_3 = -22/M, \quad \tilde{a}_4 = -12/M, \quad \alpha = 1/M, \]

where \( M \) is a sufficiently large number.