AUTOMORPHISMS IN GENERALIZED SPACES

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A survey of results on the theory of automorphisms in classical spaces with connections, Finsler spaces, and spaces of support elements are presented.

Introduction

In the present survey of results on the theory of automorphisms in generalized spaces, papers reviewed in the reviewing journal "Matematika" for the period from 1966 to October, 1977 are included.

Many interesting results in the theory of automorphisms were obtained during the period indicated. The author attempts to isolate the most important directions of investigation and to elucidate in more detail only the basic results in each of these directions.

In connection with the strengthened development of the general theory of connections in fibered spaces, we have isolated in a separate section the papers on the automorphisms of such spaces.

In a survey paper, essentially one is not concerned with questions of mappings of spaces and automorphisms in spaces with specific structures of special forms (contact, almost complex, etc.), although in the bibliography some papers from these domains of investigation are included. Automorphisms in spaces with structures are dealt with in part in the surveys of D. V. Beklemishev [Itogi Nauki "Geometry. 1963," Moscow (1965)] and A. P. Shirokov (Itogi Nauki "Algebra. Topology. Geometry. 1967," Moscow (1969) [152]). The numbers of papers from the bibliography of the preceding survey [51] are distinguished by asterisks.

1. Automorphisms in Spaces with Affine Connections.

Projective Transformations

1. The basic problem of the theory of automorphisms of generalized spaces is the problem of the distribution of gaps and segments of possible orders (segments of condensation) of the full automorphism groups, the definition for these segments of the groups themselves, and the lacunary spaces corresponding to them.

A space is said to be of the k-th lacunarity if the order of its full group of automorphisms belongs to a segment of condensation having the number k (the count is carried out with segments containing the maximal number of parameters). In particular, a space of the first lacunarity is a space of maximal mobility; a space will be of the last lacunarity if the order of its full automorphism group precedes only one gap. Essentially, the basic problem has to do with definitions of spaces and full automorphism groups under the assumption that the orders of the latter are sufficiently large.

In the case of spaces with affine connections, the segments of condensation containing maximal orders consist of just one point; the corresponding space is the ordinary affine space with automorphism group \( G_\alpha \) of order \( r = n^2 + n \), where \( n \) is the dimension of the space.

With increasing dimension the space enlarges as does the length of the interval of prohibited degrees of mobility. The length of the first prohibited interval (first gap) is equal to \( n - 1 \). For sufficiently high dimension of the space, along with the first gap, there will appear another gap. As the number \( n \) grows, the number of gaps also grows. It is not hard to verify that the second gap has length \( n - 2 \); the length of the third gap is equal to \( n - 3 \). For spaces with affine connections only part of the intervals of possible orders of the group of automorphisms is known. A description of the lacunary spaces and their tensorial characteristics is given in [48]. The global structure of maximally mobile nonflat spaces with affine connection is determined by the universal covering of the group of automorphisms. In the case considered there exist three types [48] of locally maximally mobile spaces. For each of these types there is defined a corresponding local Lie group of automorphisms. The center of the algebra of these groups of automorphisms in each of the indicated cases reduces to the zero vector.

Thus, the topological structure of the universal covering group $G_n^2$ is determined by the injective exponential map of the adjoint representation on a linear group of nondegenerate matrices.

The possible structures are obtained in the standard way by factorization of the covering obtained by a discrete central normal subgroup.

Spaces with affine connection of the second lacunarity are cosets of the group of automorphisms possible for each type by its stationary subgroup $H_{n^2-n}$, which brings us in a finite number of steps to three types of spaces with the absolutes indicated in [51].

In a series of papers Vrăinceanu [388-394] investigated the global space $A_n$ with affine connection; he extended the study of the equivalence of a locally euclidean space with a euclidean one under the assumption that the components of the affine connection $\Gamma_{jk}^i$ are entire functions. In the case when these functions are constant, one can associate with the space a commutative and associative algebra of hypercomplex numbers if and only if the connection is locally euclidean. In this case there exists a unit $e_1$, such that $e_pe_{p-1} = e_p (p \leq n)$, $e_1e_p = \Gamma_{1p}^1e_1 + \ldots + \Gamma_{np}^p \cdot A_n$ is equivalent with euclidean space if and only if $p = n$. $\Gamma_{1n}^d = 0$.

In [390] Vrăinceanu proved that in the affine space $A_3$ there exist five types of discrete groups formed by parallel transports and linear transformations $T$. whose characteristic equation has two imaginary roots.

To questions of the global structure of a space with affine connection, Vrăinceanu brought investigations of automorphisms and point transformations [288-308*].

In [390], Rozenfel’d and Tyurina made it clear that spaces with affine connection of the first lacunarity are also quasielliptic or quasihyperbolic spaces and have almost hyperplane structures; the group of motions of the space can be represented by linear-fractional transformations on the corresponding commutative algebra with unitary matrices (the elements of which belong to the algebra).

2. The investigation of properties of spaces with affine connections were extended to those of the second and third lacunarities. In [150] Shapiro considered the group of affine homotheties of the space $A_n$ with a torsion-free affine connection. A monomial group of motions is called a group of homotheties if the orbit of any geodesic is a completely geodesic surface.

Another generalization of infinitesimal homotheties of ordinary affine space on a manifold equipped with a complete linear symmetric connection $\Gamma$ belongs to Kerbrat [236]. The vector field $X$ of an infinitesimal homothety is determined as a field such that $D_YX = Y$ for any $Y$, where $D_Y$ is the symbol for covariant differentiation along $Y$ in the sense of $\Gamma$. This field has a unique critical point and has the following property: if $X$ is an infinitesimal affine transformation, then the connection is trivial.

3. In the joint paper [71], Kantor, Sirota, and Solodovnikov considered one class of symmetric spaces. Let $M$ be a symmetric space with affine connection, $G(M)$ be the group generated by translations along geodesics. In the general case this group is maximal in the sense that it is not included in a Lie group of transformations of the space $A_n$ which is larger in dimension, other than the group of motions. In the paper a construction is given of a class of symmetric spaces with extendable group $G(M)$.

In [31] Germanov considered automorphisms in reducible spaces with affine connection depending on the rank of the Ricci tensor in the summands.

Tarina [374] gives a new proof of a series of known results with respect to spaces with affine connections admitting fields of parallel contravariant or covariant vectors. It is proved that the space $A_n$, admitting $n - p$ fields of parallel contravariant vectors has group of motions depending on not more than $n^2 + (1 - p)n + p$ parameters. In [374] generalized subprojective spaces of the indicated type are considered. In another paper [373], maximal motions of a space with affine connection admitting $m$ invariant Pfaffian forms are studied; in the general case the spaces considered have transformation groups depending on $m + (n - m)m$ parameters and $n - m$ functions. Dumitras [200] is also devoted to the same question. We also note Dumitras [198, 199] on spaces $A_n$ admitting a group of homotheties of congruences, and spaces with affine connection with separable group of congruences admitting rotations.

In [31] In Shao-tzi investigates the class of homogeneous spaces with affine connection such that the isotropy group at some point admits $q$ invariant vectors and an $(n - q)$-dimensional conjugate invariant plane.

4. Yamaguchi and Matsumoto [413] concerned themselves with questions of infinitesimal affine transformations in $A_{K^*}$-spaces, i.e., in non-Riemannian $n$-dimensional spaces with torsion-free affine connection with recurrent curvature tensor (and nonzero recurrence vector $K_p$). Takano posed the question of the study