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We consider nonaxisymmetric vibrations of a hollow sphere made of a spherically isotropic material. The original boundary-value problem is reduced to a system of ordinary differential equations by introducing new unknown functions and expanding the sought solution in a series in spherical functions. The nonaxisymmetric problem of vibrations of a hollow ball decomposes into two independent problems that describe modes of first and second kind. The frequency spectrum for modes of first and second kind is analyzed.

The frequency spectrum of the eigenmodes of an isotropic ball (both solid and hollow) has been qualitatively studied in considerable detail; detailed quantitative information is also available [1, 3]. For the case of a spherically isotropic material, however, the general problem is unsolvable in special functions; axisymmetric vibrations of a hollow ball are studied in [2, 4] by expanding the displacements in Legendre polynomials and subsequently solving the ordinary differential equations in power series. In this paper, we consider the general case of nonaxisymmetric vibrations.

In the spherical coordinate system $r, \theta, \varphi$ related to the Cartesian variables by the dependences $x = r \sin \theta \cos \varphi, y = r \sin \theta \sin \varphi, z = r \cos \theta$, harmonic vibrations of a transversally isotropic ball (the surfaces $r = \text{const}$ are the isotropy surface) are described by the standard system of equations (the time factor $\exp(-i\omega t)$ has been omitted)

\[
\begin{align*}
\frac{\partial \sigma_r}{\partial r} + \frac{2\sigma_r - \sigma_\theta - \sigma_\varphi}{r} + \frac{1}{r} \frac{\partial \sigma_\theta}{\partial \theta} + \frac{\sigma_\theta}{r \cot \theta} + \\
+ \frac{1}{r \sin \theta} \frac{\partial \sigma_\varphi}{\partial \varphi} + \rho \omega^2 u_r = 0, \quad \frac{\partial \sigma_\theta}{\partial \theta} + \frac{3}{r} \sigma_\theta + \\
+ \frac{1}{r} \frac{\partial \sigma_\theta}{\partial \theta} + \frac{\sigma_\theta - \sigma_\varphi}{r} \cot \theta + \frac{1}{r \sin \theta} \frac{\partial \sigma_\varphi}{\partial \varphi} + \rho \omega^2 u_\theta = 0, \\
\frac{\partial \sigma_{\varphi}}{\partial r} + \frac{3}{r} \sigma_{\varphi} + \frac{1}{r} \frac{\partial \sigma_{\varphi}}{\partial \theta} + \frac{2}{r} \sigma_{\varphi} \cot \theta + \frac{1}{r \sin \theta} \frac{\partial \sigma_{\varphi}}{\partial \varphi} + \rho \omega^2 u_{\varphi} = 0,
\end{align*}
\]

\[
\begin{align*}
\sigma_r &= c_{11} \varepsilon_r + c_{12} (\varepsilon_\theta + \varepsilon_\varphi), \quad \sigma_\theta = c_{12} \varepsilon_r + c_{13} \varepsilon_\theta + c_{22} \varepsilon_{\varphi}, \quad \sigma_{\varphi} = c_{13} \varepsilon_r + c_{12} \varepsilon_\theta + c_{23} \varepsilon_{\varphi}, \\
\sigma_{\theta} &= c_{44} \frac{1}{r} \left( \frac{1}{r \sin \theta} \frac{\partial u_\theta}{\partial \varphi} + \frac{\partial u_{\varphi}}{\partial \theta} - u_\varphi \cot \theta \right), \\
\sigma_{\varphi} &= c_{45} \left( \frac{\partial u_{\varphi}}{\partial r} - \frac{u_\varphi}{r} + \frac{1}{r \sin \theta} \frac{\partial u_\theta}{\partial \varphi} - \frac{\partial u_{\varphi}}{\partial \theta} \right), \quad \sigma_{\theta\varphi} = c_{56} \left( \frac{\partial u_{\varphi}}{\partial r} - \frac{u_\varphi}{r} \right), \\
\varepsilon_r &= \frac{1}{r \sin \theta} \frac{\partial u_\theta}{\partial \varphi} + \frac{u_\varphi}{r} \cot \theta + \frac{u_\varphi}{r}, \\
\varepsilon_\theta &= \frac{1}{r \sin \theta} \frac{\partial u_\theta}{\partial \varphi} + \frac{u_\varphi}{r} \cot \theta + \frac{u_\varphi}{r}.
\end{align*}
\]

These equations can be simplified by the following technique [5]. Represent the displacement components $u_\theta$ and $u_{\varphi}$ by the formulas

\[
\begin{align*}
u_\theta &= \frac{\partial u_\theta}{\partial \theta} + \frac{1}{\sin \theta} \frac{\partial u_\varphi}{\partial \varphi}, \\
u_{\varphi} &= \frac{1}{\sin \theta} \frac{\partial u_\theta}{\partial \varphi} - \frac{\partial u_\varphi}{\partial \theta}.
\end{align*}
\]
in terms of the new unknown functions \( u_1 \) and \( u_2 \). Then from the defining relationships for the stresses \( \sigma_{\theta \theta} \) and \( \sigma_{r \theta} \), we obtain similar dependences

\[
\sigma_{\theta \theta} = \frac{\partial u_{\theta}}{\partial \theta} + \frac{1}{\sin \theta} \frac{\partial u_{\phi}}{\partial \phi}, \quad \sigma_{r \theta} = \frac{1}{\sin \theta} \left( \frac{\partial u_{\theta}}{\partial \phi} + \frac{\partial u_{\phi}}{\partial \theta} \right),
\]

(3)

where

\[
\sigma_1 = c_{55} \left( \frac{\partial u_1}{\partial r} - \frac{u_1}{r} \right), \quad \sigma_2 = c_{55} \left( \frac{\partial u_2}{\partial r} - \frac{u_2}{r} \right).
\]

(4)

Using formulas (2) and (3), the stress-strain relationship for \( \sigma_r \) is easily written in the form

\[
\sigma_r = c_{11} \frac{\partial u_r}{\partial r} + c_{12} \left( \frac{2}{r} u_r + \frac{1}{r} \Delta_{\theta,\phi} u_1 \right),
\]

(5)

and the first equation in (1) is rewritten as

\[
\frac{\partial \sigma_r}{\partial r} + \frac{2}{r} \sigma_r - 2c_{12} \frac{u_r}{r} + \rho \omega^2 u_r - \frac{c_{22} + c_{33}}{r^2} (2u_r + \Delta_{\theta,\phi} u_1) + \frac{1}{r} \Delta_{\theta,\phi} \sigma_1 = 0,
\]

(6)

where the differential operator

\[
\Delta_{\theta,\phi} = \frac{\partial^2}{\partial \theta^2} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2}.
\]

More complicated and tedious transformations of the second and third equations in (1) produce the relationships

\[
\frac{\partial \Phi_1}{\partial \theta} + \frac{1}{\sin \theta} \frac{\partial \Phi_1}{\partial \phi} = 0, \quad -\frac{\partial \Phi_2}{\partial \theta} + \frac{1}{\sin \theta} \frac{\partial \Phi_2}{\partial \phi} = 0.
\]

(7)

Here \( \Phi_1 \) and \( \Phi_2 \) stand for the expressions

\[
\Phi_1 = \frac{\partial \sigma_1}{\partial r} + \frac{3}{r} \sigma_1 + \rho \omega^2 u_1 + 2c_{44} \frac{u_1}{r^2} + \frac{c_{22}}{r^2} \Delta_{\theta,\phi} u_1 + c_{12} \frac{1}{r} \frac{\partial u_r}{\partial r} + \frac{c_{22} + c_{33}}{r^2} u_r,
\]

\[
\Phi_2 = \frac{\partial \sigma_2}{\partial r} + \frac{3}{r} \sigma_2 + \rho \omega^2 u_r + c_{44} \left( \frac{2}{r^2} u_2 + \frac{1}{r^2} \Delta_{\theta,\phi} u_2 \right).
\]

Since (7) holds for

\[
\frac{\partial \sigma_1}{\partial r} + \frac{3}{r} \sigma_1 + \rho \omega^2 u_1 + 2c_{44} \frac{u_1}{r^2} + \frac{c_{22}}{r^2} \Delta_{\theta,\phi} u_2 + c_{12} \frac{1}{r} \frac{\partial u_r}{\partial r} + \frac{c_{22} + c_{33}}{r^2} u_r = 0,
\]

\[
\frac{\partial \sigma_2}{\partial r} + \frac{3}{r} \sigma_2 + \rho \omega^2 u_2 + c_{44} \left( \frac{2}{r^2} u_2 + \frac{1}{r^2} \Delta_{\theta,\phi} u_2 \right) = 0,
\]

(8)

the first three equations in system (1) can be replaced with Eqs. (6) and (8). Substituting (4) and (5) in these equations, we finally obtain the system

\[
\begin{align*}
\frac{c_{11}}{r} \left( \frac{\partial^2 u_1}{\partial r^2} + \frac{2}{r} \frac{\partial u_1}{\partial r} \right) + \left[ \rho \omega^2 - \frac{2}{r^2} (c_{22} + c_{33} - c_{12}) + \frac{c_{22}}{r^2} \Delta_{\theta,\phi} u_1 \right] \frac{\partial u_1}{\partial r} + \left[ c_{12} + c_{15} \frac{\partial}{\partial r} - \frac{c_{23} + c_{33} - c_{12}}{r^2} \right] \Delta_{\theta,\phi} u_1 &= 0, \\
\frac{c_{11}}{r} \left( \frac{\partial^2 u_2}{\partial r^2} + \frac{2}{r} \frac{\partial u_2}{\partial r} \right) + \left[ \rho \omega^2 - \frac{2}{r^2} (c_{44} - c_{33}) + \frac{c_{22}}{r^2} \Delta_{\theta,\phi} u_2 \right] \frac{\partial u_2}{\partial r} + \left[ c_{12} + c_{15} \frac{\partial}{\partial r} - \frac{c_{23} + c_{33} - c_{12}}{r^2} \right] \Delta_{\theta,\phi} u_2 &= 0.
\end{align*}
\]

(9)