We consider minimax estimation of a linear functional of a homogeneous random field. A linear optimal estimator of the functional is derived and the field achieving the minimax error of the functional estimate is determined.

In this paper, we consider minimax estimation of the linear functional

\[ A^\xi_s = \int_0^\infty \int_0^\infty a(s, t) \xi(s, t) \, ds \, dt \]

of a homogeneous random field \( \xi(s, t), (s, t) \in \mathbb{R}^2 \), in the class of random fields satisfying the conditions

\[ M^\xi_s(s, t) = 0, \quad M|\xi(s, t)|^2 = 1, \]

from observations of the field at the points of the set \( \mathcal{E} = \mathbb{R}^2/\mathbb{R}^+ \). We find the form of the field which gives the largest mean-square error of the optimal linear estimator of the functional \( A^\xi \). This is a one-sided moving average field and it is defined by the continuous function \( a(s, t) \). We consider the problem subject to the following constraints:

\[ \int_0^\infty \int_0^\infty |a(s, t)| \, ds \, dt < \infty, \]

\[ \int_0^\infty \int_0^\infty s \, |a(s, t)|^2 \, ds \, dt < \infty. \]

Such problems for random fields and sequences were previously considered in [2, 5, 6] and for discrete fields in [8].

1. Evaluation of the Minimax Error for \( A^\xi_{ST} \)

Consider our problem subject to the constraints \( a(s, t) = 0, (s, t) > (S, T); S, T \in \mathbb{Z}_+ \). In this case, the functional

\[ A^\xi_{ST} = \int_S^T \int_0^\infty a(s, t) \xi(s, t) \, ds \, dt. \]

We consider linear estimators of the functional \( A^\xi_{ST} \) of the form

\[ \hat{A}^\xi_{ST} = \int_{-\infty}^S \int_0^\infty c(s, t) \xi(s, t) \, ds \, dt + \int_S^T \int_0^\infty c(s, t) \xi(s, t) \, ds \, dt, \]

where \( c(s, t) \) is an unknown complex-value function. Find the maximum value of \( \rho_{ST} = M|A^\xi_{ST} - \hat{A}^\xi_{ST}|^2 \), which determines the error of the estimator (4) for a given function \( c(s, t) \). The maximum is over all homogeneous random fields that satisfy the conditions (1). To this end, we approximate the function \( a(s, t) \) by a step function \( a_{mn}(s, t) = a(p/m, q/n) = a_{pq} \) for \( (s, t) \in \mathbb{Z}_+ \).
\[
[p/m, (p + 1)/m \times [q/n, (q + 1)/n]. \text{Since the function } a(s, t) \text{ is continuous and } M|\xi(s, t)|^2 = 1, \text{ the relationship}
\]
\[
M|A_{ST}\xi - A_{ST}^{(mn)}\xi|^2 = M\left\| \int_0^S \int_0^T a(s, t) \xi(s, t) \, ds \, dt \right\|^2
\]
\[
= \left\| \sum_{s=0}^{S} \sum_{t=0}^{T} a_{mn}(s, t) \xi(s, t) \, ds \, dt \right\|^2 = \left\| \sum_{s=0}^{S} \sum_{t=0}^{T} a(s, t) - a_{mn}(s, t) \xi(s, t) \, ds \, dt \right\|^2 \to 0
\]
as \, n \to \infty \text{ shows that substitution of } A_{ST}^{(mn)}\xi = \int_0^S \int_0^T a_{mn}(s, t) \xi(s, t) \, ds \, dt \text{ for } A_{ST}\xi \text{ produces an arbitrarily small error.}

Denote by \( \Lambda \) the class of all linear estimators of the functional \( A_{ST}^{(mn)}\xi \) from field observations at the points of the set \( E \), by \( \Lambda_1 \) the class of all linear estimators of the functional \( A_{ST}^{(mn)}\xi \) of the form (4), and by \( \Lambda_2 \) the class of all linear estimators of the functional \( A_{ST}^{(mn)}\xi \) formed from the values of the random field

\[
\xi(p, q) = \sum_{s=0}^{S} \sum_{t=0}^{T} \xi(s, t) \, ds \, dt
\]
of the form

\[
\hat{A}_{ST}^{(mn)}\xi = \sum_{p=0}^{S-1} \sum_{q=0}^{T-1} c(p, q) \xi(p, q) + \sum_{p=-\infty}^{\infty} \sum_{q=0}^{T-1} c(p, q) \xi(p, q).
\]

Moreover, denote by \( \xi \) the class of all homogeneous random fields that satisfy condition (1), by \( \xi_D \) the class of all discrete homogeneous random fields of the form (5), and by \( \xi_E \) the class of fields with \( M\xi(m, n) = 0 \) and \( M|\xi(m, n)|^2 = 1 \). Then we can write

\[
\min_{\Lambda} \max_{\xi} M|A_{ST}^{(mn)}\xi - \hat{A}_{ST}^{(mn)}\xi|^2 \leq \min_{\Lambda} \max_{\xi_D} p_{ST}^{(mn)} \leq \min_{\Lambda_1} \max_{\xi_1} p_{ST}^{(mn)} = \min_{\Lambda_2} \max_{\xi_2} p_{ST}^{(mn)}
\]

Using the spectral expansion of the random field and its correlation function [1], we obtain

\[
p_{ST}^{(mn)} = M\left\| \sum_{p=0}^{S-1} \sum_{q=0}^{T-1} \frac{1}{mn} a(p, q) \xi(p, q) \right\|^2 = \left\| \sum_{p=-\infty}^{S-1} \sum_{q=-\infty}^{T-1} b(p, q) \xi(p, q) \right\|^2
\]
\[
\times \sum_{\lambda=0}^{S} \sum_{\rho=0}^{T} e^{i(\lambda(p-\rho)+\rho(q-\gamma))}dF(\lambda, \rho) = \sum_{\lambda=0}^{S} \sum_{\rho=0}^{T} b(\lambda, \rho) \xi(p, q) dF(\lambda, \rho).
\]

Here

\[
b(\lambda, \rho) = \sum_{p=0}^{S-1} \sum_{q=0}^{T-1} \frac{1}{mn} a(p, q) e^{i(\lambda p+\rho q-\gamma)} - \sum_{\lambda=0}^{S-1} \sum_{\rho=0}^{T-1} c(p, q) e^{i(\lambda p+\rho q-\gamma)}
\]
\[
- \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{T-1} c(p, q) e^{i(\lambda p+\rho q-\gamma)}.
\]

Seeing that \( \xi(p, q) \in \xi_E \), we obtain

\[
\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} dF(\lambda, \rho) = M|\xi(p, q)|^2 = 1.
\]