OPTIMAL RATE OF INTEGRATION AND \( \varepsilon \)-ENTROPY OF A CLASS OF ANALYTIC FUNCTIONS

B. D. Boyanov

The author considers a class \( F \) of analytic functions real in the interval \([-1, 1]\) and bounded in the unit circle. As an estimate of the optimal quadrature error \( R(n) \) over the class \( F \) it is shown that

\[
\varepsilon \left( \frac{2^{2^n + 1}}{2} \right) \leq R(n) \leq \varepsilon \frac{\pi}{2^n}. 
\]

With the additional condition that \( \max_{x \in [-1, 1]} |f(x)| < \varepsilon \), an estimate is obtained for the \( \varepsilon \)-entropy \( H_{\varepsilon}(F) \):

\[
\frac{8}{2^m} \left( \ln 2^n \right) \leq \lim_{\varepsilon \to 0} \frac{H_{\varepsilon}(F)}{\varepsilon} \leq \frac{2}{\pi^2} \left( \ln 2^n \right). 
\]

Let \( F \) be the class of all functions real in the interval \([-1, 1]\) for which there exists a bounded analytic continuation of constant 1 in the circle \( G \), \( G = \{ z; |z| \leq 1 \} \). In many practical problems one has to investigate functions of this class. Here, we shall confine ourselves to estimates of the optimal quadrature over \( F \) and the \( \varepsilon \)-entropy of some subspace of \( F \).

1. Optimal Quadrature

Let us consider the problem of the approximate evaluation of the integral \( I(f) = \int_{-1}^{1} f(x) \, dx \) using the quadrature formula

\[
I(f) \approx \sum_{j=1}^{n} (C_{j} f(x_{j}) + D_{j} f'(x_{j})).
\]

We introduce the notation

\[
\omega(\vec{x}) = \inf_{\{C_{j}, D_{j}\} \in F} \max_{x_{j} \in [-1, 1]} \left| I(f) - \sum_{j=1}^{n} (C_{j} f(x_{j}) + D_{j} f'(x_{j})) \right|,
\]

\[
R(n) = \inf_{-1 < x_1 < \ldots < x_n < 1} \omega(\vec{x}).
\]

From the results of [1] and [2], it follows that

\[
\omega(\vec{x}) = \inf_{f \in F} \int_{-1}^{1} f(x) \, dx, \quad f(x_{j}) = f'(x_{j}) = 0 \quad (j = 1, 2, \ldots, n).
\]

Repeating the arguments used in [3] for the solution of a similar extremal problem, we can obtain

\[
\omega(\vec{x}) = \prod_{j=1}^{n} \left( \frac{x - x_{j}}{1 - x_{j}} \right)^{2} \, dx.
\]

Using this expression, we obtain lower and upper estimates for the optimal quadrature error over \( F \).


© 1974 Consultants Bureau, a division of Plenum Publishing Corporation, 227 West 17th Street, New York, N. Y. 10011. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, microfilming, recording or otherwise, without written permission of the publisher. A copy of this article is available from the publisher for $15.00.
The Lower Estimate. Let us first introduce some auxiliary propositions.

**LEMMA 1.** Let \(-1 < \theta < L < 1\), and let \(p\) be any real number in \((-1, 1)\). Then

\[
I(\theta, L) = \int_0^L \ln \left| \frac{x - \theta}{1 - x^2} \right| \frac{dx}{1 - x^2} \geq -\frac{\pi^2}{4}.
\]

**Proof.** We make the change of variable \(x = (t+1)/(t-1)\) in the interval \(I(\theta, L)\). Then, \(x\) is replaced in the interval \([-1, 1)\) by the new variable \(t \in [0, \infty)\). The interval \(I(\theta, L)\) takes the form

\[
I(0, L) = \int_0^L \ln \left| \frac{t + \xi}{t - \xi} \right| \frac{1}{t} dt,
\]

where

\[
\xi = \frac{1 + p}{1 - p}, \quad \theta_1 = \frac{1 - p}{1 + p} \left( \frac{1 + L}{1 - L} \right), \quad L_1 = \frac{1 - p}{1 + p} \left( \frac{1 - L}{1 + L} \right).
\]

Since \(L_1 \geq \theta_1 > 0\), then by Newman's lemma [4] we obtain \(I(\theta, L) \geq -(\pi^2/4)\).

**LEMMA 2.** Let \(-1 < x_1 \leq x_2 \leq \ldots \leq x_n < 1\), and let \(\delta \in [0, \exp \left(-\frac{\pi}{\sqrt{2}} V^n\right)]\). Then, there exists a point \(x_0 \in [-1 + \delta, 0]\) for which

\[
\left| \prod_{j=1}^n \frac{x_j - x_0}{1 - x_0^2} \right| > \delta.
\]

**Proof.** Let us assume that there exists no point in \([-1 + \delta, 0]\) for which inequality (1) holds. Then

\[
\sum_{j=1}^n \ln \left| \prod_{j=1}^n \frac{x_j - x}{1 - x_j^2} \right| \frac{1}{1 - x^2} dx \leq \ln \delta \sum_{j=1}^n \frac{1}{1 - x_j^2} dx = \frac{1}{2} \ln \delta \ln \left( \frac{2}{\delta} - 1 \right).
\]

On the other hand, by Lemma 1

\[
\sum_{j=1}^n \ln \left| \prod_{j=1}^n \frac{x_j - x}{1 - x_j^2} \right| \frac{1}{1 - x_j^2} dx = \sum_{j=1}^n \left( \prod_{j=1}^n \frac{x_j - x}{1 - x_j^2} \right) \frac{1}{1 - x^2} dx > -\frac{\pi^2}{4} n.
\]

These inequalities lead to a contradiction for \(\delta \in [0, \exp \left(-\frac{\pi}{\sqrt{2}} V^n\right)]\). The lemma is proved.

**LEMMA 3.** Let \(0 < \Delta < 1\). For every \(x \in [-1 + \Delta, 1 - \Delta]\) the inequality

\[
\left| \prod_{j=1}^n \frac{x_j - x}{1 - x_j^2} \right| \leq \frac{1}{\Delta}
\]

holds.

This lemma is an elementary consequence of Cauchy's formula.

**THEOREM 1.** For every natural number \(n\) the estimate

\[
R(n) \geq \exp \left\{ - \left( \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) \pi V^n \right\}
\]

holds.

**Proof.** By Lemma 2 there exists a point \(x_0 \in [-1 + \exp \left(-\frac{\pi}{\sqrt{2}} V^n\right), 0]\) for which \(P^n(x_0) = \left( \prod_{j=1}^n \frac{x_0 - x_j}{1 - x_0^2} \right)^n \geq e^{-\frac{\pi^2}{2} V^n}\). By Lemma 3 for all \(x \in [-1 + \exp \left(-\frac{\pi}{\sqrt{2}} V^n\right), 0]\)

\[
\left| \left( \prod_{j=1}^n \frac{x_j - x}{1 - x_j^2} \right)^n \right| \leq 2 \exp \left( \frac{\pi}{\sqrt{2}} V^n \right).
\]

552