REARRANGEMENTS OF SERIES IN $L_p$

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In this note we show that the condition

$$
\left( \sum_{n=1}^{\infty} f_n^{p} \right)^{1/p} \leq L_p
$$

is sufficient for the set of sums of the rearranged series $\sum_{\sigma} f_n$ to be a closed linear set in $L_p$.

Kadets [1] has proved that if the series

$$
\sum_{n=1}^{\infty} f_n^{p} < \infty \quad (1 \leq p \leq 2)
$$

of elements of the space $L_p$ is convergent, then the set $\mathcal{R}$ of all functions

$$
\Phi_{\alpha} \rightarrow \sum f_n
$$

which are obtained by rearranging the terms of the series ($\sum_{\sigma}$ denotes that after the sign $\sum$ the terms of the series are taken in the order $\sigma$) is a closed linear set in $L_p$. This means that $F_1 \in \mathcal{R}$ and $F_2 \in \mathcal{R}$ imply that

$$
\lambda F_1 + (1 - \lambda) F_2 \in \mathcal{R} \quad (-\infty < \lambda < \infty).
$$

For $p > 2$, condition (1) is replaced by the condition

$$
\sum_{n=1}^{\infty} \|f_n\|_p^p < \infty.
$$

We have investigated in [2] the extent to which conditions (1) and (3) are necessary conditions. The purpose of this note is to show that condition (1) can be replaced by a weaker condition. We remark that for $p = 1$ the condition (1) of Kadets is too strong:

$$
\sum_{n=1}^{\infty} \|f_n\|_1 \leq \infty.
$$

For series in $L_1$ with the condition (4) the assertion of the theorem is trivial.

The following theorem holds.

**THEOREM 1.** Let $1 \leq p \leq 2$ and assume that for the series

$$
\sum_{n=1}^{\infty} f_n
$$

the following condition is fulfilled:

$$\sqrt{\sum_{n=1}^{\infty} f_n(x)} \in L_p.$$ 

Then, the set $\mathcal{F}$ of the sums is a closed linear set in $L_p$.

The conclusion of the theorem follows from the following lemmas when one uses a construction given in [3].

We assume that for the numbers $a_1, a_2, \ldots, a_N$ and the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \ldots & N \\ n_1 & n_2 & n_3 & \ldots & n_N \end{pmatrix}$$

the number

$$S_\sigma = \sup_{1 \leq k \leq N} \left| \sum_{p=1}^{k} a_{n_p} \right|$$

is defined.

**LEMMA 1.** The following inequality holds:

$$\frac{1}{N!} \sum_{\sigma} S_\sigma^2 \leq C \left[ \sum_{k=1}^{N} a_k^2 + \left( \sum_{k=1}^{N} a_k \right)^2 \right],$$

(5)

where $C$ is an absolute constant.

The proof of Lemma 1 is a generalization of the proof of the corresponding lemma of Garcia [4].

Every permutation $\sigma$ can be uniquely decomposed into the product of the cycles $C_1, C_2, \ldots, C_{l(\sigma)}$.

We put

$$X_{C}(\sigma) = \left\{ \begin{array}{ll} 1, & \text{if } \sigma \text{ contains cycle } C, \\ 0, & \text{otherwise.} \end{array} \right.$$ 

The notation $|C| = l$ means that the cycle $C$ consists of $l$ elements.

By following the reasoning of Garcia [4], we find that

$$\frac{1}{N!} \sum_{\sigma} S_\sigma^2 \leq \frac{2}{N!} \sum_{\sigma} \left[ \sum_{C} \left| \sum_{i \in C} a_i \right| X_{C}(\sigma) \right]^2.$$

By squaring the contents of the curly brackets, we are able to split the sum in the right-hand member into the two parts

$$J_1 = \frac{2}{N!} \sum_{\sigma} \sum_{C} \left| \sum_{i \in C} a_i \right|^2 X_{C}(\sigma),$$

$$J_2 = \frac{2}{N!} \sum_{\sigma} \sum_{C_1 \neq C_2} \left| \sum_{i \in C_1} a_i \right| \left| \sum_{i \in C_2} a_i \right| X_{C_1}(\sigma) X_{C_2}(\sigma).$$

We first estimate $J_1$.

When we take into account that

$$\sum_{C} X_{C}(\sigma) = (N - |C|)!,$$

we find that

$$J_1 = \frac{2}{N!} \sum_{n=1}^{N} (N - n)! \left| \sum_{i \in C_n} a_i \right|^2.$$

We note that any collection $i_1 < i_2 < \ldots < i_l$ can form a cycle of length $l$. Hence, because the number of cycles that can be formed from the numbers $i_1 < i_2 < \ldots < i_l$ is $(l-1)!$, we obtain