


AVERAGED DESCRIPTION OF A FLUID CONTAINING GAS BUBBLES

A. L. Berdichevskii

UDC 532.529

An averaged equation is derived that describes the irrotational flow of an ideal incompressible fluid due to the expansion and translational motion of bubbles at the sites of a periodic lattice. The calculation of the coefficients of the averaged equation reduces to the solution of a cell problem. An exact solution to the problem is constructed in the form of a series in periodic harmonic functions. An infinite system of equations is written down for the coefficients of the series, and the system is analyzed asymptotically at low volume concentrations c of the bubbles.

Formulation of the Problem

We consider at a certain time a three-dimensional region \( V \) and in it gas bubbles \( A_\alpha \) at the sites \( x_\alpha \) of a periodic lattice with periods \( b_{\tau_1}, b_{\tau_2}, b_{\tau_3} \) (\( b \) is the maximal step of the lattice, \( x_\alpha \) are the Cartesian coordinates of site \( \alpha \), where \( \alpha \) is an integral vector). The bubbles \( A_\alpha \) are spheres of radius \( a_\alpha \) moving translationally with velocity \( u_\alpha \) and expanding with velocity \( \dot{a}_\alpha \); \( a_\alpha, \dot{a}_\alpha \), and \( u_\alpha \) are slowly varying functions of the number \( \alpha \).

The flow potential \( \Phi(x) \) of an ideal incompressible fluid minimizes the functional

\[
I(\Phi) = \int_V \left( \frac{1}{2} \frac{\partial \Phi}{\partial x_i} \frac{\partial \Phi}{\partial x_i} d^3x + \sum_\alpha \right) \int_{\delta A_\alpha} \Phi(x)(\dot{a}_\alpha + u_\alpha v_i) d^2x - \int_V \Phi(x) h(x) d^3x
\]  

(1)

Here, \( d^3x \) and \( d^2x \) are elements of volume and area, respectively, \( v_1 \) is the unit normal on the surface \( \delta A_\alpha \) of bubble \( A_\alpha \) exterior to \( A_\alpha \), \( h(x) \) is the normal component of

the fluid velocity on the boundary ∂V in the region V; the Latin indices take the values 1, 2, 3 and summation is understood over repeated upper and lower indices.

Because of the complicated geometry of the region \(V - \sum A_a\), it is virtually impossible to find the flow potential \(\Phi(x)\). To construct an approximate solution, we carry out an averaging.

Averaging of the functional (1) in the absence of expansion of the bubbles was considered in [1]. For expanding bubbles, the averaging can be done similarly.

**Averaging**

We divide the region \(V\) into identical cells \(B_\alpha\) (\(B_\alpha\) is the parallelepiped spanned by the three vectors \(b_1, b_2, b_3\) with center at the center of the region \(A_\alpha\)), and we consider the region \(B - A\), where \(B\) is the parallelepiped with center at the coordinate origin and generating vectors \(\tau_1, \tau_2, \tau_3\), and \(A\) is a sphere of radius \(a\) (\(a \equiv a_\alpha/b\)), also with its center at the origin. We shall denote the volumes of the regions \(B_\alpha, B, A, \) etc., by \(|B_\alpha|, |B|, |A|\), respectively.

It is well-known [2] that the solution \(\Phi(x)\) to the problem of minimizing the functional (1) to order \(o(b/\ell)\) (\(\ell\) is the characteristic dimension of the region \(V\)) can be represented in the form

\[
\Phi(x) = \varphi(x) + \psi(y, x), \quad y = x/b
\]

Here \(\varphi(x)\) and \(\psi(y, x)\) are slowly varying functions of \(x\) defined with respect to \(x\) in the whole of \(V\), and \(\psi(y, x)\) is periodic in \(y\) (\(y\) takes values within \(B - A\)).

We shall assume that the mean value of \(\psi\) over the region \(B - A\) is zero:

\[
\langle \psi \rangle_{B-A} = \frac{1}{|B-A|} \int_{B-A} \psi \, dy = 0
\]

Note that fulfillment of the restriction (3) can be achieved by appropriate redefinition of the functions \(\varphi(x)\) and \(\psi(y, x)\).

We represent the integral over \(V - \sum A_a\) in Eq. (1) as a sum of integrals over the regions \(B_\alpha - A_\alpha\). Regrouping the terms and using the circumstance that \(\varphi(x)\) and \(\varphi, i \equiv \partial \varphi/\partial x^i\) to small terms of order \(b\) are constant within each cell, we obtain

\[
I = \sum_{a} \left\{ J_a + c u^i, \varphi, i + \varphi \delta_a \varphi, i \frac{4\pi a^2}{|B_a|} \right\} |B_a| - \int_{\partial V} \psi h \, d^2 x, \quad c(x_a) = |A_a|/|B_a|
\]

\[
J = \frac{1}{2|B|} \int_{B-A} (\psi, \psi + \psi_i \psi_i^* d^2 y + \frac{1}{|B|} \int_{\partial A} (\psi + \psi_i c_i) d^2 y, \quad \psi_i = \psi^* = \psi / \partial y^i, \quad \psi_i = \psi^i
\]

The number \(\alpha\) in (5) is omitted, since it has the same form in all cells, and the dependence of the functional \(J\) on the coordinates \(x_\alpha\) is realized only through the quantities \(\varphi, i(x_\alpha), \varphi_\alpha, \varphi, a, c(x_\alpha)\), which are fixed constants within the cell \(B\).

In the limit \(b \to 0\), the sum in (4) becomes an integral and the functional \(I\) takes the form

\[
I = \int_V (J + c u^i \varphi, i + \varphi \delta_a \varphi, i) d^3 x - \int_{\partial V} \psi h d^2 x
\]

Here, \(u^i(x), c(x), a(x), \delta(x)\) are smooth interpolations of the functions \(u^i_{\alpha}, c(x_\alpha), a_\alpha/b, \delta_\alpha = a_\alpha^2/|B_\alpha|\), respectively, [3].

We seek a minimum of the functional \(I\) as follows. We first find the minimum of \(J\) over all periodic functions \(\psi\) satisfying the restrictions (3). It is obvious that the minimal value \(J^*\) of the functional \(J\) is a quadratic form in \(u^i, \varphi, i, \) and \(\delta\). Substituting the expression for it in Eq. (6), we obtain the averaged functional \(\langle I\rangle\), and minimizing it for given \(u^i(x), \delta(x),\) and \(a(x)\) we can find the mean potential \(\varphi(x)\).

Thus, the averaging problem reduces to finding the minimal value of the functional \(J\).