Thermoelastic processes in viscoelastic media, taking into account the gradients of the thermal field and of the chemical potential

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Using the methods of continuum mechanics and the thermodynamics of nonequilibrium processes a system of equations is obtained for the quantitative description of thermomechanical processes in rheologically simple single-component solids; the gradients of the thermal and chemical potentials are taken into account.

The description of thermoelastic processes in deformable solid bodies usually follows from assumptions about the applicability of the principle of local thermodynamic equilibrium [2, 3]. In [1] a generalization of this principle is derived from an assumption of a local quasiequilibrium state; as state parameters, in addition to the stress tensor \( \sigma \) or the deformation tensor \( \varepsilon \), the gradients of the temperature \( \text{grad} T \) and of the chemical potential \( \text{grad} \mu \) are applied to the temperature \( T \) or the entropy \( s \). In the framework of a model of a rheologically simple thermoelastic body [5] with parameters characterizing the system at an actual moment of time \( t \), we have \( \sigma \) or \( \varepsilon \), \( T \) or \( s \) and the history of their changes.

Here we adopt a thermodynamic approach to establishing the systems of equations for a quantitative description of interrelated mechanical, thermal, and diffusion processes in rheologically simple one-component thermoelastic bodies taking into account the gradient of the thermal fields and of the chemical potential.

We consider a thermoviscoelastic body, occupying at an actual moment of time a region \( (V) \) of euclidean space and a bounded surface \( (\partial V) \) [at the initial moment of time \( t = 0 \) it occupies the region \( (V_0) \) with surface \( (\partial V_0) \)]. The body is influenced by surface forces \( \sigma_s \) and volume forces \( F = -\rho \text{ grad} \mu - s \text{ grad} T \) (\( \rho \) is the mass density) under conditions of heat-mass-volume exchange with the external medium at a constant temperature equal to the initial temperature \( T_0 \) of the body.

The system of equations describing the behavior of the body consists of the balance equations for energy, mass, momentum, and entropy, which in Lagrangian coordinates \( \{\xi^i\} \) \( (i = 1, \ldots, 3) \) relative to an initial geometric configuration \( (V_0) \) have the form [1]

\[
\frac{\partial u_0}{\partial t} + \rho \frac{\partial}{\partial t} \left( \frac{\sigma^2}{2} \right) = \nabla \cdot \left( \sigma_0 \cdot v - TJ^{0,0} - \mu_0 J^{0,0} \right) + F_0 \cdot v,
\]

\[
\frac{\partial \rho_0}{\partial t} + \nabla \cdot J^{0,0} = 0,
\]

\[
\rho_0 \frac{\partial v}{\partial t} = \nabla \cdot \sigma_0 + F_0,
\]

\[
\frac{\partial s_0}{\partial t} + \nabla \cdot J^{0,0} = \sigma_0, \quad \nabla = \partial_i \frac{\partial}{\partial \xi^i}.
\]

Here \( J^{0,0}, J^{0,0}_M \) are the entropy and mass flows which may be represented as the sum of inverse and noninverse components:

\[
J^{0,0}_s = -\frac{\partial u_0}{\partial \tau} \cdot J^{0,0}_s, \quad J^{0,0}_M = -\frac{\partial u_0}{\partial \tau} \cdot J^{0,0}_M,
\]

\( u_0 \) is the density of the internal energy; \( v \) is the velocity vector; \( \sigma_0 \) is the first Pioli–Kirchhoff stress tensor; \( F_0 = -\rho_0 \nabla \mu - s_0 \nabla T \) is the entropy production; \( \sigma_0 \) are the contravariant basis vectors. Here and below the index "0" indicates that the values introduced are relative to the initial configuration.
The equation of energy balance, using the balance equations for mass, momentum and entropy and introducing the free-energy density \( f_0 \) with relations [1]

\[
i_0 = u_0 - T s_0 - \Pi_0^\varepsilon \cdot \nabla T - \Pi_0^M \cdot \nabla \mu
\]

can be put into the form

\[
\frac{\partial \rho}{\partial t} = - \dot{S}_0 \frac{\partial T}{\partial t} + u_0 \frac{\partial \rho_0}{\partial t} + \hat{\gamma}_0 \cdot \frac{\partial \rho_0}{\partial t} - \Pi_0^\varepsilon \cdot \frac{\partial (\nabla T)}{\partial t} - \Pi_0^M \cdot \frac{\partial (\nabla \mu)}{\partial t} -
\]

\[
- T \sigma_0^\varepsilon - \dot{J}_0^\varepsilon \cdot \nabla T - \dot{J}_0^M \cdot \nabla \mu,
\]

where \( \hat{\gamma}_0 = \partial \gamma_0 \), \( \gamma_0 \) are the covariant basis vectors for the Lagrangian system of coordinates in an actual configuration \( \mathbf{V} \).

In the sequel we take as a measure of deformation

\[
\dot{E} = \dot{\gamma}_0 \gamma_0 = \hat{I},
\]

where \( \hat{I} \) is a unit tensor.

For rheologically simple materials it is usual to introduce in place of the actual time \( t \) a fictive time \( \xi \) satisfying

\[
\xi = \int_0^t z \left( \frac{\gamma}{\gamma_0} \right) d\gamma.
\]

The shift function \( z \) might depend on the temperature \( T \), the deformation \( E \), the gradient of the temperature \( \nabla T \), the gradient of the chemical potential \( \nabla \mu \), and the rates of change for these.

In the space of our fictive time variable equation (3) has the form

\[
\frac{\partial \rho_{\xi}}{\partial \xi} = - \rho_{\xi} \frac{\partial T_{\xi}}{\partial \xi} + u_{\xi} \frac{\partial \rho_{\xi}}{\partial \xi} + \hat{\gamma}_{\xi} \cdot \frac{\partial \rho_{\xi}}{\partial \xi} - \Pi_{\xi}^\varepsilon \cdot \frac{\partial ( \nabla T_{\xi} )}{\partial \xi} -
\]

\[
- \Pi_{\xi}^M \cdot \frac{\partial ( \nabla \mu_{\xi} )}{\partial \xi} - \dot{J}_{\xi}^\varepsilon \cdot \nabla T_{\xi} - \dot{J}_{\xi}^M \cdot \nabla \mu_{\xi}.
\]

Here

\[
\frac{\partial \rho_{\xi}}{\partial \xi} = \frac{\partial \rho_0}{\partial T} \frac{\partial T_{\xi}}{\partial \xi} + \frac{\partial \rho_0}{\partial \nabla T} \frac{\partial \nabla T_{\xi}}{\partial \xi} + \frac{\partial \rho_0}{\partial \nabla \mu} \frac{\partial \nabla \mu_{\xi}}{\partial \xi} + \frac{\partial \rho_0}{\partial \mu} \frac{\partial \mu_{\xi}}{\partial \xi} + \frac{\partial \rho_0}{\partial \gamma_0} \frac{\partial \gamma_{\xi}}{\partial \xi} + \frac{\partial \rho_0}{\partial \gamma_0} \frac{\partial \gamma_{\xi}}{\partial \xi}.
\]

In order to obtain deterministic equations it is necessary to concretize the free energy \( f_0 \). We take this to be a function of the mass density \( \rho_0 \), the temperature \( T_0 \), the deformation \( E_0 \), the gradient of the temperature \( \nabla T_0 \), the gradient of the chemical potential \( \nabla \mu_0 \), and the history of their changes \( \rho_{\xi}, T_{\xi}, E_{\xi}, \nabla T_{\xi}, \nabla \mu_{\xi}, \rho_0, T_0, E_0, \nabla T_0, \nabla \mu_0 \) (0 \( \leq \xi \leq \xi_0 \)), that is,

\[
f_{\xi} = \int_{\gamma=0}^{\xi} \left( \rho_{\xi}, T_{\xi}, E_{\xi}, \nabla T_{\xi}, \nabla \mu_{\xi}, \rho_0, T_0, E_0, \nabla T_0, \nabla \mu_0 \right).
\]

Then we can obtain the following relations from Eq. (6)

\[
\left( \frac{\partial T_{\xi}}{\partial \xi} \right) + \left( \frac{\partial \nabla T_{\xi}}{\partial \xi} + \nabla \mu_{\xi} \right) \rho_{\xi} = \left( \frac{\partial T_0}{\partial \xi} \right) + \left( \frac{\partial \nabla T_0}{\partial \xi} + \nabla \mu_0 \right) \rho_0 = \left( \frac{\partial T_{\xi}}{\partial \xi} \right) + \left( \frac{\partial \nabla T_{\xi}}{\partial \xi} + \nabla \mu_{\xi} \right) \rho_0 = \left( \frac{\partial T_0}{\partial \xi} \right) + \left( \frac{\partial \nabla T_0}{\partial \xi} + \nabla \mu_0 \right) \rho_0 = 0.
\]