INTERACTIONS IN AN INFINITE MEDIUM OF PLANAR CRACKS WITH COMPLETELY RIGID PLANAR INCLUSIONS

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UDC 539.3

The problem of determining the interactions in an infinite medium of planar cracks with absolutely rigid inclusions leads to a system of integral equations, the regular kernels of which represent the interaction. The system of integral equations is completely determined under boundary conditions for the equilibrium of the inclusions as a rigid body. An approximate solution for the system of integral equations is used. The dependence of the magnitude of the external load on parameters characterizing the distribution in the medium of disc-shaped cracks and inclusions is graphically presented.

The interaction of cracks with completely rigid inclusions in a two-dimensional problem is examined in [1]; in three dimensions, the problem of the interaction of planar cracks and that of the interaction of rigid inclusions are examined separately in [3] and [5], respectively. References 3 and 5 apply diverse mathematical methods for solving problems in the theory of elasticity, which significantly complicates their application to research on the interaction in an infinite medium of cracks and rigid inclusions.

In this work integral-equation methods are used to investigate the interaction of planar cracks and rigid planar inclusions in an infinite medium. We graphically represent the dependence of the limiting load on the distribution parameters for the circular cracks and inclusions.

Let an infinite medium, containing a system M of arbitrarily distributed nonintersecting completely rigid planar inclusions of zero thickness, and a system N of planar nonintersecting arbitrarily distributed cracks be subjected to a given stretching force of intensity Q defined on an infinite domain and a force $T_{3n}$ ($j = 1, \ldots, 3$, $n = 1, \ldots, N$) defined on the surface of the cracks (Fig. 1). The forces $T_{3n}$ are taken to be normal, whereas $T_{1n}$ and $T_{2n}$ are tangential to the surfaces of the n-th crack. The problem of analytically investigating the interaction of cracks with inclusions leads to the solution of the Lamé equations under prescribed boundary conditions at infinity, on the surfaces of the cracks and at the location of the inclusions.

In order to determine the stress–deformation state of the investigated medium we choose local cartesian coordinates $O_{n}X_{1n}X_{2n}X_{3n}$ ($n = 1, \ldots, N + M$) such that the planes of the nth crack or the n-th inclusion coincide with the coordinate plane $x_{n1}O_{n}X_{2n}$. The regions occupied by the cracks will be denoted by $S_{n}$ ($n = 1, \ldots, N$), and the regions occupied by the inclusions will be denoted by $\Omega_{k}$ ($k = 1, \ldots, M$). We assume that opposite surfaces $S_{n}^\pm$ of the n-th crack correspond to values $x_{3n} = \pm 0$.

The distribution of cracks and inclusions in the medium is determined from the distances $d_{kn}$, by the direction cosines $e_{kn}$ ($j = 1, \ldots, 3$) of the vector $d_{kn}$ connecting points $O_{k}$ and $O_{n}$ (see Fig. 1), and also by the direction cosines $l_{kn}$, $m_{kn}$ and $n_{kn}$ of the coordinate axes; these are given by the matrix

$$
\begin{pmatrix}
 X_{1k} & X_{2k} & X_{3k} \\
 X_{1n} & l_{1kn} & l_{2kn} & l_{3kn} \\
 X_{2n} & m_{1kn} & m_{2kn} & m_{3kn} \\
 X_{3n} & n_{1kn} & n_{2kn} & n_{3kn}
\end{pmatrix}
$$
During the deformation of the medium there is a mutual displacement of opposite surfaces of the cracks; this displacement is characterized by functions \( \alpha_{jn}(j = 1, ..., 3, n = 1, ..., N) \). Stresses in the medium as the points cross the completely rigid inclusions possess jumps described by functions \( p_{jk}(j = 1, ..., 3, k = 1, ..., M) \). Initially taking the functions \( \alpha_{jn} \) and \( p_{jk} \) to be arbitrary, we determine the displacements and stress in the medium by means of these functions. Then, satisfying the boundary conditions of the problem, we obtain a system of integral equations for determining \( p_{jk} \) and \( \alpha_{jn} \).

If as a basis for the system of coordinates we choose the coordinate system \( O_{m}X_{1m}X_{2m}X_{3m} \), then the displacement \( U_{jm} \) (\( j = 1, 2, 3 \)) in this coordinate system can be expressed in the form

\[
U_{jm}(X_{hm}) = \sum_{n=1}^{N} \left[ d_{jm}^{(1)}(x_{hm}^{*}) + \sum_{i=1}^{M} \left[ d_{jm}^{(2)}(x_{hm}^{*}) l_{jm}^{(i)} + d_{jm}^{(3)}(x_{hm}^{*}) m_{jm}^{(i)} + d_{jm}^{(3)}(x_{hm}^{*}) n_{jm}^{(i)} \right] \right] + \sum_{k=1}^{M} \left[ d_{jk}^{(1)}(x_{hm}^{*}) l_{jk}^{(i)} + d_{jk}^{(2)}(x_{hm}^{*}) m_{jk}^{(i)} + d_{jk}^{(3)}(x_{hm}^{*}) n_{jk}^{(i)} \right]. \quad (j = 1, 2, 3),
\]

where \( d_{jm}^{(1)} \) is the displacement in the \( m \)-th system of coordinates caused by stretching of the analogous solid body by the forces \( Q; \) \( d_{jm}^{(2)} \) are the displacements in the \( l \)-th local coordinate system caused by the opening of the cracks; \( d_{jm}^{(3)} \) are the displacements in the \( k \)-th local coordinate system caused by jumps in the stresses \( p_{jk} \) which model the inclusions; \( x_{hm}^{*} \) is an arbitrary point of the medium with coordinates \( (x_{1m}, x_{2m}, x_{3m}) \) in the \( m \)-th coordinate system; \( x_{hm}^{*} \) is the same point in the \( l \)-th local coordinate system, i.e.,

\[
x_{1m}^{*} = e_{1m}^{*} d_{1m} + \sum_{i=1}^{3} l_{1m}^{(i)} x_{1m}, \quad x_{2m}^{*} = e_{2m}^{*} d_{2m} + \sum_{i=1}^{3} m_{2m}^{(i)} x_{2m}, \quad x_{3m}^{*} = e_{3m}^{*} d_{3m} + \sum_{i=1}^{3} n_{3m}^{(i)} x_{3m}.
\]

In the sequel a point with coordinates \( (x_{1m}, x_{2m}, 0) \) will be denoted by \( x_{m} \), and the same point in coordinates \( (x_{1m}^{*}, x_{2m}^{*}, x_{3m}^{*}) \) will be denoted by \( x_{m}^{*} \). Analogously, coordinates \( x_{jm}^{*} \) with \( x_{3m} = 0 \) will be denoted by \( x_{jm} \).

The displacements \( u_{jm}^{(1)} \) will be assumed known, since they are easily determined by methods described in the literature. The displacements \( u_{jm}^{(2)} \) and \( u_{jm}^{(3)} \), in accordance with [2, 4], are determined by the expressions where

\[
\begin{align*}
u is the Poisson coefficient.