FREE OSCILLATION OF A CURVILINEAR ROD

A. V. Dubovik, M. F. Kopytko, and Ya. G. Savula

A method and finite-element algorithm are presented for determining the frequencies and form of free oscillations of elastic curvilinear rods with finite rigidity under deflection. Numerical examples are introduced in order to investigate the dependence of the precision of the calculated frequencies and the form of the oscillations on the method of division into finite elements.

We examine a thin elastic rod, the axis of which is determined by the space curve

\[ x = x(\alpha), \quad y = y(\alpha), \quad z = z(\alpha), \quad (1) \]

where \( \alpha \) is an arbitrary parameter, \( \alpha \in [\alpha_N, \alpha_K] \).

We choose at the axis of the rod (1) a natural frame \( \hat{\tau}, \hat{\nu}, \hat{\beta} \) where \( \hat{\tau} \) is the tangent, \( \hat{\nu} \) is the normal, and \( \hat{\beta} \) is the binormal \( [4] \).

We introduce a static coordinate system \( \overline{\tau}, \overline{\nu}, \overline{\beta} \) where the axes \( \overline{\nu}_1 \) and \( \overline{\beta}_1 \) are directions along the major axes of the cross-sectional inertia. Let \( \delta^* \) be the angle between the axes \( \overline{\nu} \) and \( \overline{\nu}_1 \) which is a given function of the parameter \( \alpha \). Then we have

\[
\begin{align*}
\overline{\nu}_1 &= \overline{\nu} \cos \delta^* + \overline{\beta} \sin \delta^*, \\
\overline{\beta}_1 &= -\overline{\nu} \sin \delta^* + \overline{\beta} \cos \delta^*.
\end{align*}
\]

The axis of the rod relative to the coordinate system \( \overline{\tau}, \overline{\nu}_1, \overline{\beta}_1 \) is characterized by the curvature \( \omega_{\overline{\nu}_1}, \omega_{\overline{\beta}_1} \), torsion \( \omega_{\overline{\tau}} \) and element of length \( A_{\tau} \) \([4]\).

As a mathematical model of the rod we choose equations for two groups of Clebsch ratios for points on the rod, taking into account the tensile strength of the axis of the rod and the shift deformations in two planes \([2, 6]\). Within the framework of the chosen model, investigations of free oscillations of curvilinear rods lead to the determination of numbers \( \omega \) for which the problem described by the matrix equation

\[
C^0_{\theta} B C^0_{\theta} \dot{U} + \omega^2 Q U = 0, \quad (3)
\]

for homogeneous boundary conditions has a nontrivial solution \( \dot{U} \). Here \( \omega \) is the angular frequency, \( \dot{U} = \{u_{\overline{\nu}_1}, u_{\overline{\beta}_1}, \omega_{\overline{\nu}_1}, \omega_{\overline{\beta}_1}, \omega_{\overline{\tau}}\} \); \( u_{\overline{\nu}_1}, u_{\overline{\beta}_1}, u_{\overline{\tau}} \) are the displacements of an arbitrary point on the axis of the rod in the direction of the axes \( \overline{\nu}_1, \overline{\beta}_1, \overline{\tau} \); \( \omega_{\overline{\nu}_1}, \omega_{\overline{\beta}_1}, \omega_{\overline{\tau}} \) is the projection of the rotation of the frame \( \overline{\tau}, \overline{\nu}_1, \overline{\beta}_1 \); \( C^0_{\theta}, C^1_{\theta} \) are matrices of differential operators of the form

\[
C^0_{\theta} := \begin{bmatrix}
0 & 0 & 0 & 1 / A_{\tau} & \frac{d}{d\alpha} & -\omega_{\overline{\tau}} & \omega_{\overline{\beta}_1} \\
0 & 0 & 0 & \omega_{\overline{\nu}_1} & 1 / A_{\tau} & \frac{d}{d\alpha} & -\omega_{\overline{\nu}_1} \\
0 & 0 & 0 & -\omega_{\overline{\beta}_1} & \omega_{\overline{\nu}_1} & 1 / A_{\tau} & \frac{d}{d\alpha} \\
\frac{1}{A_{\tau}} & \frac{d}{d\alpha} & -\omega_{\overline{\tau}} & \omega_{\overline{\beta}_1} & 0 & -1 & 0 \\
-\omega_{\overline{\tau}} & -\frac{1}{A_{\tau}} & \frac{d}{d\alpha} & \omega_{\overline{\nu}_1} & -1 & 0 & 0 \\
-\omega_{\overline{\beta}_1} & \omega_{\overline{\nu}_1} & \frac{1}{A_{\tau}} & \frac{d}{d\alpha} & 0 & 0 & 0
\end{bmatrix}
\]


0090-4104/93/6504-1786$12.50 ©1993 Plenum Publishing Corporation
\[ C = \begin{bmatrix}
0 & 0 & 0 & \frac{1}{A_x} & \frac{d}{d\alpha} & \omega_\tau & \omega_{\beta} \\
0 & 0 & 0 & \omega_\tau & -\frac{1}{A_x} & \frac{d}{d\alpha} & -\omega_{\nu} \\
0 & 0 & 0 & -\omega_{\beta} & -\omega_{\nu} & \frac{1}{A_x} & \frac{d}{d\alpha}
\end{bmatrix} \]

B is a diagonal 6 x 6 matrix, the elements of which are \( b_{ij} = \delta_{ij} g_i \), where \( \delta_{ij} \) is the Kronecker delta; \( g_1 \) and \( g_2 \) denote the rigidity of the rod with respect to twisting about the axis corresponding to \( v_1 \) and \( \bar{v}_1 \); \( g_3 \) is the rigidity of the rod with respect to torsion; \( g_4 \) and \( g_5 \) denote the rigidity with respect to displacement in the planes \( (v_1, \bar{v}, (\bar{v}_1, \bar{v}) \); \( g_6 \) is the rigidity of the rod with respect to stretching. Expressions for \( g_i \) are given in [5]; \( Q \) is a diagonal 6 x 6 matrix, on the diagonal of which are inertial terms of the form \( Q_{11} = Q_{22} = Q_{33} = \rho S, Q_{44} = \rho l_{v_1}, Q_{55} = \rho l_{\bar{v}_1}, Q_{66} = \rho (l_{v_1} + l_{\bar{v}_1}) \), where \( \rho \) is the density; \( S \) is the area of a section; \( l_{v_1}, l_{\bar{v}} \) are moments of inertia relative to the axis.

In the variational formulation [1], problem (3) for homogeneous boundary conditions leads to the determination of \( \omega^2 \) and the vector \( \vec{U} \neq 0 \) satisfying the relation

\[ \sum_{\alpha} (C^{\alpha})^{\top} B C^{\alpha} \vec{V} A_{\alpha} d\alpha - \omega^2 \int_{\alpha_N} Q \vec{V} A_{\alpha} d\alpha = 0, \]

where \( \vec{V} \) is an arbitrary element of the space \( H_1 \times H_1 \times H_1 \times H_1 \times H_1 \times H_1 \) satisfying homogeneous boundary conditions.

We represent the domain of definition of the parameter \( \alpha, \Omega = [\alpha_N, \alpha_K] \) as a union of \( n \) segments \( \Omega_e = [\alpha_{e-1}, \alpha_e] \).

The unknown vector function \( \vec{U} \) will be approximated by quadratic functions. To do this we distinguish for each element a node which divides an element \([\alpha_{e-1}, \alpha_e] \) in half

\[ \alpha_{e+, 1/2} = \frac{\alpha_e + \alpha_{e+1}}{2}. \]

Performing an isoparametric transformation of the region \( \Omega_e \) to \( \Omega_e = [-1 \leq \xi \leq 1] \) such that the point \( \alpha_{e+, 1/2} \) is represented by the point \( \xi = 0 \), we write

\[ \vec{U}_e = H(\xi) \vec{q}_e, \]

where \( H(\xi) \) is a block matrix of the form

\[ H(\xi) = [\Phi_1, \Phi_2, \Phi_3]. \]

\( \Phi_1 \) is a diagonal 6 x 6 matrix, along the diagonal of which are quadratic functions \( \varphi_i(\xi) \):

\[ \varphi_1 = \frac{1}{2} (\xi - 1) \xi, \varphi_2 = 1 - \xi^2, \varphi_3 = \frac{1}{2} (\xi + 1); \]

\[ \vec{q}_e = [\vec{q}_e, \vec{q}_e, \vec{q}_e]; \vec{q}_e \] is the vector of unknown nodal displacements at the j-th node of the e-th finite element.

Applying a known mean-square procedure [1, 7], we obtain a generalized eigenvalue problem

\[ K\vec{q} = \omega^2 M\vec{q} = 0. \]