In [1] it was proved that the unit ball of a reflexive Banach space has an uncountable set of extreme points. In this note it is shown that this set is also massive in some topological sense.

In [1] the following theorems are proved.

**THEOREM A.** If $E$ is an infinite-dimensional reflexive Banach space, then the set of extreme points of its unit ball is uncountable.

**THEOREM B.** Let $E$ be a Banach space and let the unit ball of the space $E^*$ have a countable set of extreme points. Then

1) $E^*$ is separable, and
2) $E$ does not contain any finite-dimensional reflexive subspaces.

In these theorems the massiveness of the set of extreme points is characterized by its cardinality.

The object of this note is to obtain a generalization of Theorems A and B in which the massiveness of the set of extreme points is characterized in topological terms.

**THEOREM 1.** Let $E$ be a Banach space and $K$ be a $ω^α$-closed, convex, and bounded subspace of $E^*$. If the set of extreme points of $K$ is "strongly" (i.e., in the norm topology of $E^*$) separable, then $K$ is the closure in the norm topology of the convex hull of its extreme points.

**Proof.** Let \( \{f_j\}_j \) be a strongly dense subset of $\text{ext } K$ and $ε_j ≜ 0$. For each natural number $n$ we denote by \( \{V^n_j\}_{j=1}^{∞} \) the sequence of closed balls of radius $ε_j$ with centers at the points $f_j$. It is clear that

$$\text{ext } K ∋ \bigcup_{j=1}^{∞} V^n_j = H^n \quad (n = 1, 2, \ldots).$$

Since each ball is a $ω^α$-closed set, $H^n$ is a set of type $F_α$. We set

$$A^n_m = V^n_m \setminus \bigcup_{i=1}^{m-1} V^n_i \quad (m, n = 1, 2, \ldots).$$

For each $n = 1, 2, \ldots$, it is obvious that we have

$$A^n_i \cap A^n_j = \emptyset \quad (i \neq j).$$

Without loss of generality, we may assume that all $A^n_m$ are nonempty. We must prove that any element $f_0 ⊂ K$ can be approximated with any degree of accuracy by convex combinations of extreme points. Since $K$ is a weak*-convex compact set, by the Choquet-Bishop-de Leeuw theorem (see [2], the note on p. 33) there exists a probability measure $μ$ which represents $f_0$ and which for any $n$ is concentrated on $H^n = \bigcup_{m=1}^{∞} A^n_m$:
We take an arbitrary \( \delta > 0 \) and take \( n \) so that \( \varepsilon_n < \delta/2 \). With the help of the measure \( \mu \) we form the series \( \sum_{m=1}^{\infty} \mu(A^n_m) f_m \) and with it we approximate \( f \):

\[
\left| f_0 - \sum_{m=1}^{\infty} \mu(A^n_m) f_m \right| = \sup_{[0,1]} \left| f_0(x) - \sum_{m=1}^{\infty} \mu(A^n_m) f_m(x) \right| = \sup_{[0,1]} \left| \int_{\mathbb{R}^n} f_0(x) \mu(dx) - \sum_{m=1}^{\infty} \int_{A^n_m} f_m(x) \mu(dx) \right| =
\]

\[
= \sup_{[0,1]} \left| \sum_{m=1}^{\infty} \int_{A^n_m} (f - f_m)(x) \mu(dx) \right| \leq \sum_{m=1}^{\infty} \sup_{[0,1]} \left| (f - f_m)(x) \right| \mu(A^n_m).
\]

Since

\[
A^n_m \subseteq V^n_m \quad \text{and} \quad f_m \equiv 1^n_m,
\]

each term of the last series satisfies

\[
\left| (f - f_m)(x) \right| \mu(A^n_m) \leq \left| f - f_m \right| \mu(A^n_m) \leq \delta \mu(A^n_m).
\]

Consequently,

\[
\left| f_0 - \sum_{m=1}^{\infty} \mu(A^n_m) f_m \right| \leq \sum_{m=1}^{\infty} \delta \mu(A^n_m) = \delta.
\]

Because \( \delta \) was arbitrary, this last inequality proves the theorem.

**COROLLARY.** Let \( E \) be a Banach space and let the set of extreme points of the unit ball of the space \( E^* \) be strongly separable. Then \( E^* \) is a separable Banach space.

**THEOREM 2.** If \( E \) is an infinite-dimensional reflexive Banach space, then the set of extreme points of its unit ball cannot be covered by a countable collection of sets which are compact in the norm topology.

**Proof.** We follow the method developed in [1]. We suppose to the contrary that

\[
\text{ext } U \subseteq \bigcup_{i=1}^{\infty} K_i,
\]

where each \( K_i \) is compact in the norm topology. Without loss of generality, we may assume that

\[
K_i \subseteq U = \{ x \in E : \| x \| \leq 1 \} \quad (i = 1, 2, \ldots).
\]

We note that it follows from the corollary to Theorem 1 that \( E \) is a separable Banach space and, consequently, \( U^* \) (the unit ball of \( E^* \)) is a metrizable compact set in the \( w^* \)-topology of \( E^* \) (which in this case obviously coincides with the \( w \)-topology). We set \( F_n = \{ f \in U^* : \exists x \in K_n, f(x) = \| f \| \} \) \((n = 1, 2, \ldots)\). We will prove that each \( F_n \) is closed in the \( w^* \)-topology of \( E^* \). Let

\[
f^{(k)} \rightharpoonup f, \quad f^{(k)} \in F_n
\]

\((n \text{ fixed}, k = 1, 2, \ldots)\). We will show that \( f \in F_n \). For each \( k \) there exists a point \( x^{(k)} \in K_n \) such that

\[
f^{(k)}(x^{(k)}) = \| f^{(k)} \|.
\]

Due to the compactness of \( K_n \) there exists a subsequence \( \{ x^{(k_i)} \}_{i=1}^{\infty} \) of the sequence \( \{ x^{(k)} \}_{k=1}^{\infty} \) and a point \( x \in K_n \) such that \( \| x^{(k_i)} - x \| \rightarrow 0 \) as \( i \rightarrow \infty \). We have

\[
\| f^{(k_i)}(x^{(k_i)}) - f(x) \| \leq \| f^{(k_i)}(x^{(k_i)}) - f^{(k)}(x) \| + \| f^{(k)}(x) - f(x) \| \leq \| x^{(k_i)} - x \| + \| f^{(k)}(x) - f(x) \| \rightarrow 0
\]

as \( i \rightarrow \infty \).

Since

\[
\| f \| \leq \lim_{k} \| f^{(k)} \| \leq \lim_{i} \| f^{(k_i)} \| = \lim_{i} f^{(k_i)}(x^{(k_i)}) = \lim_{i} f^{(k)}(x^{(k_i)}) = f(x),
\]

we have \( f(x) = \| f \| \). Consequently, \( f \in F_n \), which also proves that \( F_n \) is \( w^* \)-closed. From the reflexivity of \( E \) and the Krein–Milman theorem it follows that

\[
U^* = \bigcup_{n=1}^{\infty} F_n.
\]