ON POTENTIAL THEORY FOR TWO-DIMENSIONAL QUASI-STATIC PROBLEMS OF UNCOUPLED THERMOELASTICITY

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The construction of potential theory for two-dimensional quasi-static problems of uncoupled thermoelasticity is carried out by considering the full system of differential equations of the problem as a nonselfadjoint differential operator. Green's second formula for this operator is interpreted as a duality theorem that differs from Mizel's duality theorem. In the case of a homogeneous isotropic medium we construct new integral equations for the basic initial-boundary value problems.

Coupling effects are neglected in the differential equations of uncoupled thermoelasticity [1, 3, 4, 6–9]. In this situation if the causes of the deformation change very slowly over time, the inertial term in the equation of motion can be omitted. As a result we arrive at the system of equations for uncoupled quasistatic problems of thermoelasticity. The author [4] has given a survey of papers devoted to the theoretical and applied studies (based on the potential method) of problems of uncoupled thermoelasticity and a brief discussion of certain of his own new results in this area. In this connection the present paper does not discuss the present state of this problem. In the present paper we discuss in detail a new potential theory for two-dimensional quasi-static problems of uncoupled thermoelasticity. The construction of such a theory is carried out by considering the full system of differential equations of the thermoelastic problem mentioned above as a nonselfadjoint differential operator. Green's second formula for this operator is regarded as a duality theorem that differs from Mizel's duality theorem. Using the duality theorem constructed we construct new integral equations for the basic initial-boundary problems in the direct formulation (in the case when homogeneous isotropic media are being considered) which are effective from the point of view of application.

The Fundamental Equations. The system of linearized differential equations for two-dimensional quasi-static problems of uncoupled thermoelasticity, as is known [6], have the following form (for inhomogeneous anisotropic media):

\[-\frac{1}{2}C_{\alpha\beta\gamma\delta}(u_{\gamma,\delta} + u_{\delta,\gamma})_{,\beta} + (b_{\alpha\beta}T)_{,\beta} = X_\alpha;\]
\[c_\varepsilon \dot{T} - (\lambda_{\alpha\beta}T_{,\beta})_{,\alpha} = G, \quad (\alpha, \beta, \gamma, \delta = 1, 2).\]  

Here \(u_\alpha\) are the components of the displacement vector, \(X_\alpha\) are the components of the mass force vector, \(G\) is the density of the heat source, \(T\) is the deviation of the current absolute temperature from the absolute temperature \(T_0\) of the undeformed state, \(C_{\alpha\beta\gamma\delta}\) are the isothermic elastic constants, \(c_\varepsilon\) is the heat capacity at constant deformation, \(\lambda_{\alpha\beta}\) are the coefficients of thermal conductivity, and the quantities \(b_{\alpha\beta}\) are connected with both the mechanical and thermal constants of the body.

Here the Duhamel-Neumann expression for the stresses has the following form:

\[\sigma_{\alpha\beta} = \frac{1}{2}C_{\alpha\beta\gamma\delta}(u_{\gamma,\delta} + u_{\delta,\gamma}) - b_{\alpha\beta}T.\]

Let us consider the system of equations (1)–(2) in the space \(E^2\) and also in \(S \subset E^2\). It is assumed that the domain \(S\) has a piecewise-smooth boundary \(L\) for \(t > 0\). Points in \(E^2\) are denoted \(x = (x_1, x_2)\) and \(y = (y_1, y_2)\). The system of equations (1)–(2) is supplemented by the initial condition

\[T|_{t=0} = T_0(x)\]
and the boundary conditions
\[
\begin{align*}
\alpha_{m}\alpha u_{\alpha} + \alpha_{m}\alpha P_{\alpha}|_{L} &= f_{m}(x, t), \\
\alpha_{1}^{2}T + \alpha_{2}^{2}\frac{\partial T}{\partial n(x)}|_{L} &= h, \quad (m = 1, 2),
\end{align*}
\]
where \( n = (n_1, n_2) \) is the unit normal to the boundary \( L \) (outward-pointing in relation to \( S \)) and
\[
P_{\alpha} = \sigma_{\alpha\beta}n_{\beta} = C_{\alpha\beta}\delta n_{\alpha} - b_{\alpha\beta}Tn_{\beta}. \tag{7}
\]
The functions \( \alpha_{m}\alpha, \alpha_{m}\alpha, \alpha_{1}, \alpha_{2} \) determine the type of the boundary conditions.

It is assumed that the tensors \( C_{\alpha\beta\gamma}, b_{\alpha\beta}, \) and \( \lambda_{\alpha\beta} \) are symmetric:
\[
C_{\alpha\beta\gamma} = C_{\gamma\alpha\beta} = C_{\beta\alpha\gamma} = C_{\alpha\beta\gamma}; \quad b_{\alpha\beta} = b_{\beta\alpha}; \quad \lambda_{\alpha\beta} = \lambda_{\beta\alpha}. \tag{8}
\]

**The Duality Theorem.** To construct the duality theorem we write the system of linear differential equations adjoint to (1)-(2) (cf. [1, 4, 9]):
\[
-\left[\frac{1}{2}C_{\alpha\beta\gamma}\delta(u_{\gamma,\delta} + u_{\gamma,\delta})\right]_{\beta} = X'_{\alpha}, \quad \epsilon T - (\lambda_{\alpha\beta}T'_{\alpha})_{\alpha} = G' + b_{\alpha\beta}\epsilon'_{\gamma,\delta}. \tag{9}
\]
The Duhamel-Neumann relations for the adjoint system have the following form:
\[
\sigma'_{\alpha\beta} = \frac{1}{2}C_{\alpha\beta\gamma}\delta + b_{\alpha\beta}\epsilon'_{\gamma,\delta}. \tag{10}
\]
We now apply the Laplace transform to the Duhamel-Neumann relations (3) and (9). As a result we obtain
\[
\sigma_{\alpha\beta} = C_{\alpha\beta\gamma}\delta - b_{\alpha\beta}T, \quad \sigma'_{\alpha\beta} = C_{\alpha\beta\gamma}\delta + b_{\alpha\beta}\epsilon'_{\gamma,\delta}. \tag{11}
\]
The equation (11) thus obtained is the first part of the duality theorem. We shall obtain the second part of the duality theorem by considering the heat equation. It has the following form:
\[
\int_{L}[\lambda_{\alpha\beta}(T'_{\alpha}T'_{\beta} - T'_{\alpha}T'_{\beta})]_{\alpha}dt - \int_{S}(G'_{\alpha} - G'_{\alpha})dS - \int_{S}c_{\alpha}\bar{T}_0dS = \int_{S}b_{\alpha\beta}\epsilon'_{\alpha\beta}T dS. \tag{12}
\]
Eliminating the integral \( \int_{S}b_{\alpha\beta}\epsilon'_{\alpha\beta}T dS \) from relations (11) and (12) and applying the inverse Laplace transform to the equation obtained, we arrive at the duality theorem for two-dimensional quasistatic problems of uncoupled thermoelasticity:
\[
\int_{S}[u_{\alpha}(x, t)X'_{\alpha}(x) - u'_{\alpha}(x)X'_{\alpha}(x, t)]dS_{x} + \int_{0}^{t}\int_{S}[G'(x, t - \tau)T'(x, \tau) - G(x, \tau)T'(x, t - \tau)]dS_{x}d\tau = \\
\int_{0}^{t}\int_{L}\left\{\lambda_{\alpha\beta}(x)\left[T'(x, t - \tau)\frac{\partial T'(x, \tau)}{\partial n(x)} - T(x, \tau)\frac{\partial T'(x, t - \tau)}{\partial n(x)}\right]\right\}dS_{x}d\tau + \int_{S}[u_{\alpha}(x)p_{\alpha}(x, t) - u_{\alpha}(x, t)p'_{\alpha}(x)]dS_{x} + \\
\int_{0}^{t}c_{\varepsilon}(x)T_{0}(x)T'(x, t)dS_{x}. \tag{13}
\]