APPLICATION OF THE VARIATIONAL-MOMENT APPROACH TO PROBLEMS OF THE THEORY OF ELASTICITY OF SHELLS

Yu. D. Zozulyak and Yu. M. Gnativ

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We consider a method of obtaining an approximate system of equations of elasticity theory for shells, based on representing the components of the stress tensor and the displacement vector as a sum of products of moment characteristics depending on the coordinates of the base surface of the shell and functions of the normal coordinate to the base surface.

The first author and Ya. I. Burak [1, 2] have proposed a variational-moment approach to the construction of an approximate system of equations of dynamic thermoelasticity of thin shells on whose boundaries a load distribution is given. In the present article this approach is modified and extended to problems in the theory of elasticity of thick-walled shells of variable thickness.

Consider the functional

$$\Pi[\sigma, u] = - \int_{(V)} \left[ \frac{1}{2} \sigma : b + u \cdot (\nabla \cdot \sigma + F) \right] dV + \int_{(\Sigma)} u \cdot (\sigma \cdot n - p_n) d\Sigma,$$

where $(V)$ is the region occupied by the body, $(\Sigma)$ is the surface of the body, $\sigma$ is the stress tensor, $u$ is the displacement vector, $b$ is the tensor of elastic flexibility of the material, $F$ and $p_n$ are respectively the vectors of mass and surface force, $n$ is the unit outward normal to the surface $(\Sigma)$, $\nabla$ is the Hamiltonian operator, $(:)$ is the operation of double scalar product. The condition for stationarity of this functional is equivalent to the relations of the linear theory of elasticity:

1. $\nabla \cdot \sigma + F = 0$ in $(V)$;

2. $\sigma : b - \frac{1}{2}(\nabla u + u \nabla) = 0$ in $(V)$;

3. $\sigma \cdot n - p_n = 0$ on $(\Sigma)$.

We place relations (2)–(4) in correspondence with an approximate system of equations that determine the stressed state of the shell referred to a mixed orthogonal coordinate system $\alpha_1, \alpha_2, \alpha_3$, where the coordinate lines $\alpha_1$ and $\alpha_2$ coincide with the lines of principal curvature of the base surface $(\Sigma_0)$, and $\alpha_3$ is the coordinate in the normal direction to the surface $(\Sigma_0)$. In this connection we give the components of the stress tensor and the displacement vector in the form

$$\sigma^{ij}(\alpha_1, \alpha_2, \alpha_3) = M^{ij}_m(\alpha_1, \alpha_2)\varphi^{(i)}(\alpha_3);$$

$$u_i(\alpha_1, \alpha_2, \alpha_3) = U_{(i)m}(\alpha_1, \alpha_2)\psi_{(i)}(\alpha_3).$$

Here and throughout $M^{ij}_m = M^{ij}_m, \varphi^{ij}_m = \varphi^{ij}_m, i, j = 1, 3, m = 1, N$. The moment characteristics $M^{ij}_m(\alpha_1, \alpha_2), U_{(i)m}(\alpha_1, \alpha_2)$, and the functions $\varphi^{ij}_m(\alpha_3)$ and $\psi_{(i)}(\alpha_3$) are assumed to be unknown quantities, the repeated indices are summation indices, but summation is not carried out over indices in parentheses.

Applying the representations (5) and (6), from the condition for stationarity of the functional (1), we find

$$\int_{(\Sigma_0)} \left\{ \left[ \frac{1}{2}(\nabla u + u \nabla) - \sigma \right] : b + (M^{ij}_m \delta \varphi^{ij}_m + \varphi^{ij}_m \delta M^{ij}_m) e_i e_j - (\nabla \cdot \sigma + F) \cdot (U_{(i)m} \delta \psi_{(i)} + \psi_{(i)} \delta U_{(i)m}) e^i \right\} H_1 d\alpha_3 d\Sigma_0.$$
\[
+ \int_{\Sigma_0} \left[ \Phi_\sigma(\delta U_{im}, \delta \psi_{im}) H_1 \right]_{\alpha^3 = (1)} + \Phi_\sigma(\delta U_{im}, \delta \psi_{im}) H_1 \right]_{\alpha^3 = (1)} \ d\Sigma_0
\]

\[
+ \int_{(\Gamma_0)} \Phi_\sigma(\delta U_{im}, \delta \psi_{im}) H_2 \ d\alpha^3 \ d\Gamma_0 = 0, \quad (7)
\]

where

\[
\Phi_\sigma(\delta U_{im}, \delta \psi_{im}) = (\sigma \cdot n - p_n) \cdot (U_{im} \delta \psi_{im} + \psi_{im} \delta U_{im}) e^i,
\]

\[
H_1 = (1 + k_1 \alpha^3)(1 + k_2 \alpha^3),
\]

\[
H_2 = [1 + 2\alpha^3(k_1 \cos^2 \lambda + k_2 \sin^2 \lambda) + (\alpha^3)^2(k_1^2 \cos^2 \lambda + k_2^2 \sin^2 \lambda)]^{1/2},
\]

and \( \alpha^1, \alpha^2 = 2h(\alpha^1, \alpha^2) \) is the thickness of the shell; \( \Gamma_0 \) is the contour that forms the boundary of \( \Sigma_0 \); \( e_i, e^i \) are the covariant and contravariant basis vectors of the coordinate system \( \alpha^1, \alpha^2, \alpha^3 \); \( k_1, k_2 \) are the principal curvatures of the surface \( \Sigma_0 \); and \( \lambda \) is the angle between the coordinate line \( \alpha^1 \) and the contour \( \sigma \Gamma_0 \).

For the relations (7) to be applicable it is necessary that the following differential equations hold:

\[
a_{s_{ms}}^{(\sigma)} \frac{\partial U_{(s \sigma)}}{\partial \alpha^{(s)}} + a_{s_{ms}}^{(\sigma)} \frac{\partial U_{(s \sigma)}}{\partial \alpha^{(s)}} - 2c_{ms}^{(\sigma \omega \omega)} U_{ks} - 2b_{klms}^{(\sigma \omega \omega \omega)} M_{s}^{kl} = 0;
\]

\[
a_{s_{ms}}^{(\sigma)} \frac{\partial U_{3s}}{\partial \alpha^{(s)}} + a_{s_{ms}}^{(\sigma)} \frac{\partial U_{3s}}{\partial \alpha^{(s)}} - 2c_{ms}^{(\sigma \omega \omega)} U_{ks} - 2b_{klms}^{(\sigma \omega \omega \omega)} M_{s}^{kl} = 0;
\]

\[
d_{s_{ms}}^{(\sigma)} U_{3s} - 33k_{ms}^{(3)} U_{ks} - 33k_{klms}^{(3)} M_{s}^{kl} = 0,
\]

\[
a_{s_{ms}}^{(i)3} \frac{\partial M_{(s \sigma)}}{\partial \alpha^{(s)}} + c_{s_{ms}}^{(i)3} M_{s}^{(i)3} + g_{(i)jms}^{(i)3} M_{s}^{(i)3} + p_{jklms}^{(i)3} M_{s}^{(i)3} - q_{(i)jms}^{(i)3} M_{s}^{(i)3} + q_{s_{ms}}^{(i)3} + f_{ms}^{(i)3} = 0,
\]

\[
a_{s_{ms}}^{(i)3} \psi_{(s \sigma)} + A_{s_{ms}}^{(i)3} \psi_{(s \sigma)} - 2C_{ms}^{(i)3} \psi_{ks} - 2B_{klms}^{(i)3} \varphi_{s}^{kl} = 0;
\]

\[
D_{s_{ms}}^{(i)3} \frac{\partial \psi_{(s \sigma)}}{\partial \alpha^{(s)}} + A_{s_{ms}}^{(i)3} \frac{\partial \psi_{(s \sigma)}}{\partial \alpha^{(s)}} - 2C_{ms}^{(i)3} \psi_{ks} - 2B_{klms}^{(i)3} \varphi_{s}^{kl} = 0;
\]

\[
D_{s_{ms}}^{(i)3} \frac{\partial \varphi_{s}}{\partial \alpha^{(s)}} - C_{ms}^{(i)3} \psi_{ks} - B_{klms}^{(i)3} \varphi_{s}^{kl} = 0,
\]

\[
D_{s_{ms}}^{(i)3} \frac{\partial \varphi_{s}}{\partial \alpha^{(s)}} + E_{s_{ms}}^{(i)3} \varphi_{s}^{(i)3} + G_{s_{ms}}^{(i)3} \varphi_{s}^{(i)3} + P_{jklms}^{(i)3} \varphi_{s}^{(i)3} + Q_{s_{ms}}^{(i)3} \varphi_{s}^{(i)3} - Q_{s_{ms}}^{(i)3} + F_{ms}^{(i)3} = 0,
\]

along with the boundary conditions

\[
V_{(i)jms}^{(i)3} = V_{ms}^{(i)3} \text{ when } \alpha^3 = -h, \ h, \ \psi_{(i)jms}^{(i)3} = \psi_{ms}^{(i)3} \text{ on } (\Gamma_0).
\]

Here

\[
a_{ms}^{i \omega} = \int_{(\Sigma_0)} \varphi_{ms}^{(i) \omega} \psi_{s}^{(i)} H_1 \ d\alpha^3; \quad b_{klms}^{ij} = \int_{(\Sigma_0)} \varphi_{ms}^{(i) \omega} b_{kls}^{ij} H_1 \ d\alpha^3; \quad c_{ms}^{ij} = \int_{(\Sigma_0)} \varphi_{ms}^{(i) \omega} c_{kls}^{ij} H_1 \ d\alpha^3;
\]

\[
d_{ms}^{i \omega} = \int_{(\Sigma_0)} \varphi_{ms}^{(i) \omega} \frac{\partial \psi_{s}^{(i) \omega}}{\partial \alpha^3} H_1 \ d\alpha^3; \quad e_{ms} = \int_{(\Sigma_0)} \varphi_{ms}^{(i) \omega} \psi_{s}^{(i) \omega} H_1 \ d\alpha^3; \quad f_{ms} = \int_{(\Sigma_0)} F_{ms}^{(i) \omega} H_1 \ d\alpha^3;
\]

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