A criterion for the stability on a finite time interval, with respect to a given measure, of parametrically excited distributed processes is obtained. To this end the comparison method is used in conjunction with Lyapunov's second method and extremal properties of the Rayleigh ratios for self-adjoint operators in Hilbert space. Sufficient conditions for technical stability with respect to a given measure are established for the problem of a fixed strut loaded by a longitudinal dynamic force.


1. A General Criterion for Technical Stability of Parametrically Excited Processes. Consider the class of dynamical systems in the domain $T_1 \times D$, where $D \subset \mathbb{R}^r$ is a domain with boundary $C$, $\mathbb{R}^r$ is the $r$-dimensional Euclidean space with the coordinate vector $x = (x_1, \ldots, x_r)$, that are described on a finite time interval $T_1 \subset T = [t_0, \infty)$ by equations of the type

$$\frac{\partial u(t, x)}{\partial t} = L(t)u(t, x), \quad x \in D, \quad t \in T_1. \quad (1)$$

Here $u(t, x)$ is a $2N$-dimensional state vector which obeys homogeneous boundary conditions

$$Gu(t, x) = 0, \quad x \in C, \quad (2)$$

and the initial condition

$$u(t_0, x) = u_0(x), \quad (3)$$

where $u_0(x)$ is a function having partial derivatives of all required orders in $D$; $L(t)$ denotes a $(2N \times 2N)$ matrix of linear partial differential operators with respect to the spatial variables with time-dependent continuous coefficients. The operator $G$ is a linear, time-independent differential operator with respect to the spatial variables. Consider the functional space $H$ of continuous $2N$-dimensional real vector-functions defined in $T_1 \times D$. For each pair $z_1, z_2 \in H$ define the inner product

$$(z_1, z_2) = \int_D z_1^*(t, x)z_2(t, x)\,dx, \quad (4)$$

where "*" denotes transposition. We shall assume that $H$ is extended in such a way that it is Hilbert space [5]. The norm in $H$ will be denoted by $\| \cdot \|_H$.

Suppose the state set $W$ of the process is a subset of $H$ such that its elements satisfy the boundary condition (2) and other necessary smoothness conditions which guarantee that the function $L(t)u$ is continuous in $T_1 \times D$ for the operator $L(t)$ of the original problem (1)-(3), acting in the domain $W$. Then to the solution $u(t, x)$ of the problem one can associate, at each time $t$, an element of the space $W$, and the solution $u(t, x)$ itself forms a trajectory in $W$. Also, $L(t)$ is an operator in $H$.
acting in the domain $W \subset H$, which we denote by $W \rightarrow H$. For $u, v \in W$ define the measure

$$
\rho (u, v) = \left[ (A_0 z, A_0 z) + \sum_{i=1}^{n} (A_i z, A_i z) \right]^{1/2},
$$

where $z = u - v$, $A_0 = I$ is the identity operator, and $A_1, \ldots, A_n$ are linear time-independent differential operators in $H$, acting in the domain $W$. For the metric $\rho (u, 0)$ we have

$$
\rho (u, 0) = (u, Mu)^{1/2},
$$

where integration by parts was used under the assumption that the resulting boundary values are equal to zero thanks to condition (2). Here the operator $M = I + \sum_{i=1}^{n} A_i A_i^*$. Consider the functional

$$
V \left[ u, t \right] = (u, B(t) u), \quad u \in W,
$$

where $B(t)$ is self-adjoint in the sense that

$$
(v, B(t) w) = (w, B(t) v), \quad v, w \in W,
$$

and may contain necessary differential elements with respect to the spatial variables; as a matrix, $B(t)$ has the same dimension as $L(t)$. Choose $B(t)$ to ensure that $V[u, t]$ will be positive-definite with respect to the measure $\rho(u, 0)$. The latter means that

$$
V[u, t] \geq \alpha \rho^2 (u, 0)\quad (9)
$$

for some constant $\alpha > 0$. Obviously, by (6), condition (9) has the form $(u, [B(t) - M]u) \geq 0, \alpha > 0$.

Let us consider problem (1)-(3) in the domain $\Omega = \{t, x, u: t \in T_1 \subset T, x \in D \subset R^n, \|u\| \leq a = \text{const} > 0, \forall u \in W \subset H, \ L(t): W \rightarrow H\}$. We shall assume that $B(T)$ has the necessary differentiability properties with respect to $t$. Let the operator $N(t): W \rightarrow H$ be defined by $N(t) = L^*(t)B(t) + B(t)L(t) + B(t)$, where $L^*(t)$ is the adjoint of the operator $L(t)$, and suppose that the eigenvalue problem

$$
N(t)u = \lambda u, \quad u \in W,
$$

has the property that its eigenvalues $\{\lambda_n(t)\}$ are bounded real quantities for all $t \in T_1$, where the time $t$ is considered as a parameter, and not as an independent variable. Denote by $\lambda_{\text{max}}(t)$ the maximal eigenvalue of problem (10). In the domain $\Lambda = \{t, y, \lambda_{\text{max}}(t): t \in T_2 \subset T, -\infty < y < \infty, |\lambda_{\text{max}}(t)| \leq b = \text{const} > 0\}$ consider a function $\Phi(t, y, \lambda_{\text{max}}(t))$ that is continuous in $\Lambda$ and vanishes at $y = 0$, and the corresponding scalar Cauchy problem:

$$
\frac{dy}{dt} = \Phi(t, y, \lambda_{\text{max}}(t));
$$

$$
y(t_0) = y_0.
$$

**Definition.** The unperturbed process $u(t, x) = 0$ corresponding to the initial boundary-value problem (1)-(3) is said to be technically stable on the finite time interval $T_1 \subset T$ with respect to the measure $\rho(u, 0)$ if along the perturbed solution $u(t, x)$ of problem (1)-(3) the functional $V[u, t]$, positive-definite with respect to the measure $\rho(u, 0)$, and with operator $B(t): H \rightarrow H$ given beforehand, satisfies the condition

$$
V[u(t, x), t] \leq P(t), \quad t \in T_1,
$$

provided only that $V[u(t_0, x), t] \leq \tilde{\lambda}$, where the function $P(t)$, defined on the interval $T_1$, satisfies the conditions

$$
P(t_0) \geq \tilde{\lambda}, \quad \bar{P} = \sup_{t \in T_1} \{P(t)\} < +\infty, \quad \tilde{\lambda} = \text{const} > 0,
$$

and $P(t)$ as well as $\tilde{\lambda}$ and $T_1$ are given beforehand.