THE SPECTRAL ASYMPTOTICS OF ELLIPTIC OPERATORS
OF SCHRÖDINGER TYPE ON A HYPERBOLIC SPACE

D. V. Efremov and M. A. Shubin

UDC 517.984

For self-adjoint second-order elliptic differential operators that satisfy the non-trapping condition on the
n-dimensional hyperbolic space $\mathbb{H}^n$ and coincide with the operator $-\Delta - \left(\frac{n-1}{2}\right)^2$ in a neighborhood of
infinity, where $\Delta$ is the Laplace-Beltrami operator on $\mathbb{H}^n$, we obtain the complete asymptotic expansion
of the spectral function as $\lambda \to +\infty$. For self-adjoint operators of the form $(-\Delta)^{m/2} + Q_{m-r}$, where
$Q_{m-r}$ is a pseudodifferential operator of order $m-r$ that is automorphic with respect to a discrete group
of isometries of the space $\mathbb{H}^n$ whose fundamental domain has finite volume, we introduce the spectral
distribution function $N(\lambda)$, which is the analog of the integrated state density, and we find its asymptotics
up to order $O(\lambda^{(n-r)/m})$ as $\lambda \to +\infty$. Bibliography: 49 titles.

The spectral theory of the Laplace operator and other elliptic operators on the hyperbolic space $\mathbb{H}^n$ and
the fundamental domains of discrete groups of isometries of this space play an important role not only in
the general theory of elliptic operators, but also in geometry and number theory (the Selberg formula and
related questions). There are numerous papers (cf., for example, [1-20]) on various aspects of this theory.
The most-studied case is the case $n = 2$, i.e., the case of the hyperbolic plane $\mathbb{H}^2$ and the fundamental
domains of its discrete isometry groups. A large number of papers, beginning with the famous paper of
Selberg [19], have been devoted to the spectral theory of the Laplace-Beltrami operator on $\mathbb{H}^2$. Of these
we note first of all the paper of Faddeev [9], in which the spectral decomposition of the Laplace operator on a
fundamental domain of finite volume in $\mathbb{H}^2$ was obtained. Pavlov and Faddeev [7] constructed a theory of
scattering for $\mathbb{H}^2$. The book of Lax and Phillips [5] is devoted to the same topic. It should be emphasized
that in all these papers the subject was operators on spaces of automorphic functions.

We note that $\mathbb{H}^n$ is the symmetric space $G/K$, where $G$ is the connected component of the identity
in the group $O(1, n)$ of orthogonal transformations of the pseudo-Euclidean space $\mathbb{R}^{1,n}$ and $K = SO(n)$
(cf. [3], Chapter 1, § 10]). The eigenfunctions of the Laplace-Beltrami operator on symmetric spaces have
been studied by Karpelevich [4]. Helgason [14] has constructed a Fourier transform for symmetric spaces
of negative curvature, making it possible to obtain the spectral decomposition of the Laplace-Beltrami
operator. Donnelly [12] has studied a finite perturbation of the metric for such spaces. Using the methods
of perturbation theory, he has studied the nature of the spectrum of the Laplace-Beltrami operator of the
perturbed metric.

The present paper consists of two parts. In the first part (§ 2-4) we study a second-order elliptic
differential operator $L$ on $\mathbb{H}^n$ that coincides outside a certain sphere with the operator $L_n = -\Delta - ((n -
1)/2)^2$, where $\Delta$ is the Laplace-Beltrami operator of the standard metric on $\mathbb{H}^n$. For the analogous operators
on $\mathbb{R}^N$ Popov and Shubin [34] and Vainberg [25; 26] obtained the complete asymptotic expansion of the
derivative of the spectral function with respect to the spectral parameter $\lambda$ as $\lambda \to +\infty$ when the non-
trapping condition holds for rays. For such operators there is a local decrease of energy for the corresponding
wave equation (cf. [49]), making it possible to find the complete asymptotics essentially by taking the inverse
Fourier transform of a fundamental solution of the wave equation. Analogous results have been obtained
here for the operator $L$ on $\mathbb{H}^n$. In this case the local decrease of energy is exponential in all dimensions
(i.e., for even values of $n$ as well as for odd values), which simplifies the reasoning in some places.

Let $e(\lambda, x, y)$ be the spectral function of the operator $L$, and suppose that one ray of the operator $L$
go from $y_0$ to $x_0$ and along it $x_0$ is noncaustic (for precise definitions see § 4). Then for $e' = \partial e/\partial \lambda$ the

Translated from Trudy Seminarov imeni I. G. Petrovskogo, No. 15, pp. 4-32, 1991. Original article submitted April
asymptotic relation
\[ e'(\lambda, x, y) \sim \sum_{j=0}^{\infty} \lambda^{\frac{n-3}{4} - \frac{1}{2}} e^{i\sqrt{\lambda} h(x,y)} c_j(x,y) \]
holds as \( \lambda \to +\infty \) provided the pair of points \((x, y)\) is near the pair \((x_0, y_0)\). If there are no bicharacteristic loops at the point \(x_0\), then for \(x\) near \(x_0\) the function \(e'\) has the asymptotics
\[ e'(\lambda, x, x) \sim \sum_{j=0}^{\infty} \lambda^{\frac{3}{2} - j - 1} d_j(x) \]
on the diagonal. This asymptotic relation can be integrated to obtain the complete asymptotic expansion of \(e(\lambda, x, x)\). These results are due to the first author and were announced in [29].

In the second half of this paper (§5–8) we consider \(\Gamma\)-invariant operators on \(H^n\), where \(\Gamma\) is a discrete subgroup of the group of isometries of the space \(H^n\) whose fundamental domain \(\mathcal{F}\) has finite volume. More precisely, let \(H = (-\Delta)^{m/2} + H_1\), where \(H_1\) is a \(\Gamma\)-invariant singular pseudodifferential operator of order \(m - r\) (\(r > 0\)) having uniform symbol estimates along the entire space \(H^n\). In particular, \(H\) may be a Schrödinger operator with bounded \(\Gamma\)-invariant potential (and then \(r = 2\)). For the operator \(H\) we introduce the spectral distribution function \(N_\Gamma(\lambda)\), which is the analog of the integrated state density, by the formula
\[ N_\Gamma(\lambda) = \int_\mathcal{F} e(\lambda, x, x) \, dx, \tag{0.1} \]
where \(dx\) is the Riemannian volume element on \(H^n\) and \(e(\lambda, x, y)\) is the spectral function of the operator \(H\). The main result of the second section is the following asymptotic formula for \(N_\Gamma(\lambda)\) as \(\lambda \to +\infty\):
\[ N_\Gamma(\lambda) = \sum_{j=0}^{k} c_j \lambda^{(n-2j)/m} + O(\lambda^{(n-r)/m}), \tag{0.2} \]
where \(k = [(r-1)/2]\). We note that if \(r > 1\), then the estimate of the remainder in (0.2) is better than that given by Hörmander (cf., for example, [42, Ch. III]) for the distribution function of the eigenvalues of an elliptic operator on a compact manifold.

In the case of the Schrödinger operator the asymptotic relation (0.2) assumes the form
\[ N_\Gamma(\lambda) = c_0 \lambda^{n/2}(1 + O(\lambda^{-1})). \tag{0.3} \]
Such an asymptotic relation for the integrated state density \(N(\lambda)\) of a Schrödinger operator with almost-periodic potential was proved by Shubin [39] using a variational principle in the \(\Pi_\infty\)-factor introduced by Coburn, Moyer, and Singer [46]. In doing this use was made of the expression found in [41] for \(N(\lambda)\) as the trace of the spectral projection \(E_\lambda(H)\) of the operator \(H\) in the \(\Pi_\infty\)-factor. We are following the same route here, using the von Neumann algebra \(\mathfrak{A}_\Gamma\), which consists of all bounded \(\Gamma\)-invariant operators in \(L^2(H^n)\) and having a canonical trace \(\text{tr}_\Gamma\) on \(\mathfrak{A}_\Gamma\) for which \(N_\Gamma(\lambda) = \text{tr}_\Gamma E_\lambda(H)\).

We note that for almost-periodic and random Schrödinger operators formula (0.3) for \(N(\lambda)\) can be obtained in an elementary manner, using the interpretation of the function \(N(\lambda)\) as an integrated state density (i.e., the limit of the distribution functions, normalized through division by the volume, of the discrete spectrum of the given operator with Dirichlet or Neumann boundary condition in a bounded domain that is expanding to infinity) [43]. However the interpretation of our function \(N_\Gamma(\lambda)\) as an integrated state density is apparently not possible, since the volume of a ball and the area of its spherical surface in \(H^n\) have the same rate of growth, and it is not clear how to choose an expanding system of domains such as was used in the definition of an integrated state density.

A formula of the form (0.2) can also be obtained for the integrated state density \(N(\lambda)\) of self-adjoint almost periodic or random elliptic operators in \(\mathbb{R}^n\) of the form \(H = P(D) + H_1\), where \(P(D)\) is an elliptic...