We determine the space that parametrizes the cone of symmetrizers of a semisimple operator with real spectrum in a finite-dimensional vector space. It is proved that the set that determines the cone of symmetrizers is parametrized by the portion of the unit sphere of the vector space lying the first octant.

The problem of symmetrizability of the linear operators of Hilbert space and the construction of a theory of symmetrizable operators seems to have arisen immediately after the development of the fundamentals of the theory of symmetric operators, as a natural generalization of it. The latter has been intensively studied by many mathematicians. The absence of criteria for existence and methods of constructing symmetrizers has caused researchers to assume their existence and develop those portions of the theory of symmetrizable operators connected with various properties of symmetrizers [2].

In practice [1] for the finite-dimensional case a criterion for symmetrizability of an operator has been established, a method of constructing symmetrizers has been given, and the full variety of symmetrizers has been described. As a corollary of these results in the present article we establish a natural parametrization of the variety of symmetrizers in \( \mathbb{R}^n \). We begin by recalling the necessary concepts, introducing notation, and stating the basic results of [1]. Let \( V \) be an \( n \)-dimensional unitary space. An operator \( A : V \to V \) is (left)-symmetrizable if there exists a Hermitian operator \( S \) such that \((SA)^* = SA\). To study the most common case we impose the condition of positive definiteness on the symmetrizer \( S \).

**Theorem 1.** An operator in a finite-dimensional unitary space is symmetrizable if and only if it is similar to a Hermitian operator, or equivalently, it is a semisimple operator with a real spectrum.

Let \( A \) be a symmetrizable operator. We represent it with respect to a certain basis of the space \( V \) by a matrix \( A \), and we bring the latter into rational canonical form using a nonsingular matrix \( T \), so that \( TAT^{-1} = L \). Here \( L \) is a block-diagonal matrix whose typical block is the companion matrix of a scalar polynomial \( f(\lambda) = \lambda^m + a_1\lambda^{m-1} + \cdots + a_m \) with simple real roots \( \lambda_1 < \cdots < \lambda_m \):

\[
L(f) : L(f)y = z, \quad z_k = y_{k+1}, \quad k = 1, m-1, \quad z_m = -\sum_{i=1}^{m} a_{m+1-i}y_i.
\]

The polynomial \( f(\lambda) \) is associated with the following matrix \( S \):

\[
S(f) : S(f)y = z, \quad z_k = \sum_{i=0}^{m-k} a_{m-k-i}y_{i+1}, \quad a_0 = 1, \quad k = 1, m,
\]

whose application will be crucial during the symmetrization.

Constructing the variety of symmetrizers for the matrix \( A \) then reduces to the analogous problem for a typical block \( L(f) \). In solving the latter problem an essential role is played by the following concept. The polynomials \( f(\lambda) \) and \( g(\lambda) = s + b_1\lambda^{m-2} + \cdots + b_{m-1} = (\lambda - \alpha_1) \cdots (\lambda - \alpha_{m-1}) \) form a positive real pair of polynomials if all of their roots are real, simple, and strictly alternating: \( \lambda_1 < \alpha_1 < \lambda_2 < \cdots < \alpha_{m-1} < \lambda_m \). We denote the set of polynomials of the form \( g(\lambda) \) that form positive real pairs with the polynomial \( f(\lambda) \) by \( \text{Prp}(f) \).

**Theorem 2.** The variety of symmetrizers of the matrix \( L(f) \) forms a convex truncated cone \( K[L(f)] \) in the space of \( m \times m \) matrices given by the set \( \text{Symm}[L(f)] = \{ S = S(f)g[L(f)] : g \in \text{Prp}(f) \} \), and for the original matrix \( A \) by the set \( \text{Symm}(A) = T^*\text{Symm}(L)T \).

Hence the problem under consideration of parametrizing the cone of symmetrizers \( K(A) \) reduces to parametrizing a typical determining set \( \text{Prp}(f) \) for an invariant polynomial of the \( \lambda \)-matrix \( \lambda I - A \). In what follows we shall assume that \( A \) is a Hermitian \( n \times n \) matrix with a simple spectrum and consider the set \( \text{Prp}(f) \), where \( f \) is the characteristic polynomial of the matrix \( A \).

Let \((\lambda I - A)^\vee\) be the classical adjoint of the \( \lambda \)-matrix \( \lambda I - A \), so that 
\[
(\lambda I - A)(\lambda I - A)^\vee = (\lambda - A)^\vee(I - A) = f(\lambda)I,
\]
where \( I \) is the \( n \times n \) identity matrix. We introduce the following set of polynomials—the Hausdorff domain of the \( \lambda \)-matrix \((\lambda I - A)^\vee\):
\[
\mathcal{H}[(\lambda I - A)^\vee] = \{g_x(\lambda) = ((\lambda I - A)^\vee x, x) : (x, x) = 1\}.
\]

We shall show that \( \text{Prp}(f) \) is a subset of \( \mathcal{H}[(\lambda I - A)^\vee] \) and is parametrized by a certain part of the unit sphere \( O \) in the space \( V \).

For simplicity we give another representation of this Hausdorff domain. The following observation serves this purpose. Since the unit sphere \( O = \{x \in V : (x, x) = 1\} \) is invariant with respect to the action of an arbitrary unitary matrix \( U \) and the conjugate action of the matrix \( U \) on the \( \lambda \)-matrix \((\lambda I - A)^\vee\) is described by the relation 
\[
U^*(\lambda I - A)^\vee U = (\lambda - U^*A)U^\vee,
\]
it follows that 
\[
\mathcal{H}[(\lambda I - U^*A)U^\vee] = \mathcal{H}[(\lambda I - A)^\vee].
\]
Choosing \( U \) to be the matrix that brings \( A \) into diagonal form, so that 
\[
U^*A = D = \text{diag}(\lambda_1, \ldots, \lambda_n),
\]
we obtain
\[
\mathcal{H}[(\lambda I - A)^\vee] = \mathcal{H}[(\lambda I - D)^\vee].
\]

We remark that under such a transition to the matrix \( D \) neither the characteristic polynomial \( f(\lambda) \) nor the set \( \text{Prp}(f) \) changes.

In practice from the representations \( \mathcal{H}[(\lambda I - D)^\vee] \) it is not difficult to determine the general form of its elements \( g_x(\lambda) \). Indeed, since 
\[
(\lambda I - D)^\vee = \text{diag}(p_1(\lambda), \ldots, p_n(\lambda)),
\]
where 
\[
p_i(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_i - \lambda)(\lambda - \lambda_{i+1}) \cdots (\lambda - \lambda_n),
\]
it follows that
\[
g_x(\lambda) = \sum_{i=1}^{n} p_i(\lambda) |x_i|^2, \quad x \in O.
\]

To determine the relative positions of the roots of the polynomials \( f(\lambda) \) and \( g_x(\lambda) \) we apply the following classical proposition.

**Lemma.** The roots of the polynomial \( g(\lambda) = \lambda^n - \cdots + b_{n-1} \) are real, simple, and alternate with the roots of the polynomial \( f(\lambda) \) if and only if the matrix \( S = S(f)g[L(f)] \) is positive definite.

This proposition is easy to establish if one applies the following congruence transformation:
\[
W^*SW = \text{diag} \{f'(\lambda_1)g(\lambda_1), \ldots, f'(\lambda_n)g(\lambda_n)\},
\]
where \( W = ||\lambda_j^{-1}||_{j=1}^{n} \) is the Vandermonde matrix formed from the roots of the polynomial \( f(\lambda) \).

Applying this test, we find that an element \( g_x(\lambda) \in \mathcal{H}[(\lambda I - D)^\vee] \) belongs to the set \( \text{Prp}(f) \) if and only if the matrix \( S_x = S(f)g_x[L(f)] \) is positive definite.

We shall now find out which subset of the sphere \( O \) determines the set \( \text{Prp}(f) \). For \( x = (x_1, \ldots, x_n) \in O \) we have
\[
W^*S_xW = \text{diag} \{f'(\lambda_k) \sum_{i=1}^{n} p_i(\lambda_k) |x_i|^2, \ldots\} = \text{diag} \{f'(\lambda_k)|x_j|^2, \ldots\},
\]
because
\[
p_i(\lambda_k) = \begin{cases} f'(\lambda_k), & i = k, \\ 0, & i \neq k. \end{cases}
\]
From this we deduce that the matrix \( S_x \) will be positive definite, and hence that the polynomial \( g_x(\lambda) \) will belong to the set \( \text{Prp}(f) \) if and only if \( x_k \neq 0 \) for all values of \( k \in \{1, \ldots, n\} \). The other condition is geometrically characterized as follows. Let \( H_i \) be the coordinate hyperplane determined by the equation \( x_i = 0 \). We form the “coordinate cross” \( H = \bigcup_{i=1}^{n} H_i \). Then this condition means that the point \( x \) belongs to the set \( O' = O - O \cap H \)—the unit sphere with the points that belong to the “coordinate cross” removed.