OPTIMIZATION OF AXISYMMETRIC THERMAL DISPLACEMENTS IN A GIVEN SECTION OF AN UNBOUNDED LAYER

V. M. Vigak, A. V. Yasinskii

We study the problem of optimal control of the distribution of axisymmetric vertical or radial thermal displacements in a given section of an unbounded layer. Using the method of the inverse problem of thermoelasticity we construct the solution of the control problem. For specific cases of heating of the layer we give a numerical analysis of the behavior of the optimal control.

The functioning of many elements of modern engineering, energy, and optical equipment subject to intense thermal action depends to a large degree on the stability of the geometry of the working surface that is determined by the thermal displacements [1–3, 6]. For that reason the problem of seeking an optimal mode of using the equipment that minimizes the thermal displacements in the boundary surface is of current practical importance.

In the present article we study the problem of optimal control of vertical axisymmetric thermal displacements in a given section of an unbounded layer free of surface forces.

Suppose the heat transfer between an isotropic homogeneous unbounded layer occupying the domain \( D = \{ (r, z) \mid r \in [0, \infty), z \in (0, 1) \} \) and the surrounding medium proceeds through the boundary surface according to Newton’s law. The stationary axisymmetric temperature field of the layer in the presence of internal heat sources is described by the solution of the heat equation

\[
\Delta T(r, z) + f(r, z) = 0, \quad (r, z) \in D,
\]

satisfying the boundary conditions

\[
\frac{\partial T(r, 0)}{\partial x} - H_1(T(r, 0) - t_1(r)) = 0, \quad r \in [0, \infty);
\]

\[
\frac{\partial T(r, 1)}{\partial x} + H_2(T(r, 1) - t_2(r)) = 0, \quad r \in [0, \infty),
\]

where \( \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \) is the Laplace operator; \( r = h\rho, z = hx, h \) is the thickness of the layer; \( \rho \) and \( x \) are dimensionless cylindrical coordinates; \( H_i \) \( (i = 1, 2) \) are the dimensionless heat transfer coefficients; \( t_i(r) \) \( (i = 1, 2) \) are the temperatures of the surrounding media; \( f(r, z) \) is the relative intensity of internal heat sources, \( f = h^2 f/\lambda; f \) is the intensity of heat sources; \( \lambda \) is the coefficient of heat conductivity.

Choosing as a control the relative intensity of internal heat sources \( f(r, z) \), we pose the following control problem; it is required to find the distribution of the function \( f \in C(D) \) at which the functional

\[
I(f) = \max_{\rho \in [0, \infty]} \left| S(\rho, x_1; f) - \varphi(\rho) \right|
\]

attains its minimum. Here \( S(\rho, x_1; f) \) are the vertical \( (u_z) \) or radial \( (u_r) \) thermal displacements of the layer in the section \( x = x_1 \) \( (x_1 = \text{const}, x_1 \in [0, 1]) \); and \( \varphi(\rho) \) is the required distribution of displacements in this section.

Assume there exists a control \( f_* \in C(D) \) at which the functional (4) attains its greatest lower bound, i.e., a solution of the equation

\[
I(f_*) = \inf_{f} I(f) = 0,
\]

which is equivalent to the equality

\[ S(\rho, x_1; f_\star) = \varphi(\rho), \quad \rho \in [0, \infty). \]  

Suppose the solution of the heat equation (1)-(3) and the corresponding thermoelastic problem for the layer has been found in closed form and represented as

\[
T(\rho, x) = \int_0^\infty \int_0^1 G(\rho, x; \xi, \eta) f(\xi, \eta) \, d\eta \, d\xi + \sum_{i=1}^2 \int_0^\infty G_i(\rho, x; \xi) t_i(\xi) \, d\xi; \quad u_i(\rho, x) = \int_0^\infty \int_0^1 P_i(\rho, x; \xi, \eta) T(\xi, \eta) \, d\eta \, d\xi, \quad i = r, z \tag{6}
\]

where \( G, G_i (i = 1, 2), \) and \( P_i (i = r, z) \) are known functions.

Using the expression (7) for the thermal displacements and the solution of the heat conduction problem (6), on the basis of (5) we arrive at an integral equation of first kind

\[
\int_0^\infty \int_0^1 K(\rho, x_1; \xi, \eta) f(\xi, \eta) \, d\eta \, d\xi = \varphi(\rho) - \sum_{i=1}^2 \int_0^\infty K_i(\rho, x_1; \xi) t_i(\xi) \, d\xi \tag{8}
\]

for determining the unknown intensity of heat sources that guarantee a prescribed distribution of thermal displacements, i.e., we arrive at the inverse problem of thermoelasticity, in which one must find heat sources from a prescribed distribution of thermal displacements. Here \( K \) and \( K_i (i = 1, 2) \) are known functions. However for a fixed value of the coordinate \( x (x = x_1) \) and a prescribed function \( \varphi(\rho) \) (as a function of one variable) it is impossible to determine the function \( f_\star \) as a function of two variables from the preceding equation. Therefore we take

\[ f_\star(\rho, x) = u(\rho)x(x), \]

where \( u(\rho) \) is the unknown function and \( x(x) \) is a known function.

Equation (8) for determining the function \( u(\rho) \) in this case has the form

\[
\int_0^\infty K(\rho, x_1; \xi) u(\xi) \, d\xi = \varphi(\rho) - \sum_{i=1}^2 \int_0^\infty K_i(\rho, x_1; \xi) t_i(\xi) \, d\xi. \tag{9}
\]

The solution of the axisymmetric problem of thermoelasticity for an unbounded layer free of surface forces can be constructed by using its representation as the thermoelastic potential of the displacements and the Love function [4]. Thus the function \( P_\star \) in the expression (7) can be written in the form

\[
P_\star(\rho, x; \xi, \eta) = \frac{(1 + \nu)\alpha_T h \xi}{2(1 - \nu)} \int_0^\infty \frac{s J_0(s p) J_0(s \xi)}{\sinh^2 s - s^2} \left[ (\sinh^2 s - s^2 \cosh s |x - \eta| \sgn (x - \eta) - 4(1 - \nu) s \sinh s(x - \eta) - (1 - 2x) s^2 \cosh s(x - \eta) + (2(1 - \nu) \sinh 2s + 2x \sinh^2 s) \sinh s(x + \eta) + ((4\nu - 3) \sinh^2 s - sx \sinh 2s) \cosh s(x + \eta) + (2s - \sinh 2s) \cosh s \eta \sinh s |x| \right] \, ds,
\]

where \( \nu \) and \( \alpha_T \) are respectively the Poisson coefficient and the coefficient of linear thermal expansion; \( J_n(s) \) is the Bessel function of first kind of order \( n \).

Substituting the solution of the heat conduction problem (1)-(3) into the dependence just found, we determine the functions \( K \) and \( K_i (i = 1, 2) \) in Eq. (9). Now applying the Hankel transform on the coordinate \( \rho \) to the integral equation so obtained [5], we represent its solution in the space of transforms as

\[
u H(s) = \frac{2}{A(s, x_1, x_0)} \left[ H(s) \, d(s) - \sum_{i=1}^2 A_i(s, x_1) H_i(s) \right], \tag{10}
\]

(for simplicity we take \( x(x) = \delta(x - x_0) \), where \( \delta(x) \) is the delta-function), where \( A, A_i (i = 1, 2), \) and \( d \) are known functions and \( \nu H(s), \varphi H(s), t_i^H(s) \) are the Hankel transforms of the functions \( \nu(\rho), \varphi(\rho), \) and \( t_i(\rho) \).