We show that a free nilpotent group of countable rank, as well as a free group of countable rank of the variety defined by the identity \([x_1, x_2, \ldots, x_n], [x_{n+1}, x_{n+2}] = 1\), satisfies the maximal condition for normal subgroups admitting endomorphisms induced by order preserving one-to-one mappings of the set of free generators into itself.

It is known [1] that free metabelian (i.e., two-step solvable) groups of countable rank satisfy the maximal condition for normal subgroups admitting automorphisms induced by permutations of the free generators.

In this note we show that a free nilpotent group of countable rank of an arbitrary nilpotence class, and also a free group of countable rank of the variety defined by the identity \([x_1, x_2, \ldots, x_n], [x_{n+1}, x_{n+2}] = 1\), satisfies the condition for normal subgroups admitting endomorphisms induced by order preserving one-to-one mappings of the set of free generators into itself.

1. Preliminary Concepts and Statements

We recall some concepts and facts from [1] and prove some preliminary statements. We say that an algebraic closure relation is defined on a set \(C\) if with every subset \(X\) of \(C\) there is associated a subset \(c/X\) of \(C\) such that: a) \(X \subseteq cX\); b) if \(X \subseteq Y\), then \(cX \subseteq cY\); c) \(c(cX) = cX\); d) if \(x \in cX\), then \(x \in cX_0\) for some finite \(X_0 \subseteq X\). For example, in a commutative ring the closed sets are ideals. An algebraic closure relation has the finite basis property (FBB) if any closed set is the closure of a finite set.

A partially ordered set is called narrow [2] (or partially well-ordered [1]) if an arbitrary infinite sequence of elements contains an increasing (not necessarily strictly increasing) subsequence.

We will need the following examples of such sets.

1. Let \(P\) be the set of all finite sequences of nonnegative integers. We define two order relations on \(P\):
   a) \((i_1, i_2, \ldots, i_m) \prec (j_1, j_2, \ldots, j_n)\) if \(m < n\) or \(m = n\) and for some \(r\) \(i_r < j_r\) but \(i_s = j_s\) for all \(s > r\);
   b) \((i_1, i_2, \ldots, i_m) \prec (j_1, j_2, \ldots, j_n)\) if \(m \leq n\) and for some order preserving one-to-one mapping \(\varphi\) of the sets \((1, 2, \ldots, m)\) into \((1, 2, \ldots, n)\) \(i_r \leq \varphi(j_r)\) for all \(r\).

   It is known that \((P, \preceq)\) is well-ordered and that the partially ordered set \((P, \prec)\) is narrow.

2. Let \(Q\) be the set of all sequences of some arbitrary finite length \(n\) of positive integers. We define two order relations on \(Q\):
   a) \((i_1, i_2, \ldots, i_n) \prec (j_1, j_2, \ldots, j_n)\) if for some \(r\) \(i_r < j_r\), but \(i_s = j_s\) for all \(s > r\);
   b) \((i_1, i_2, \ldots, i_n) \prec (j_1, j_2, \ldots, j_n)\) if for some order preserving one-to-one mapping \(\varphi\) of the set of integers \(J\) into itself \(\varphi(i_r) = (j_{\varphi(r)})\) for all \(r = 1, 2, \ldots, n\). We will denote the subgroup of all order-preserving one-to-one mappings of \(J\) into itself by \(\Phi\).

   It is clear that \((Q, \preceq)\) is well-ordered.
Lemma 1. The partially ordered set \((Q, \triangleleft)\) is narrow.

Proof. Divide the elements of \(Q\) into classes as follows. Let \(a = (i_1, i_2, \ldots, i_n)\) be an element of \(Q\) for which
\[
i_{k_1} \leq i_{k_2} \leq \cdots \leq i_{k_n},
\]
and place all elements of \(Q\) whose corresponding components satisfy the same inequalities in the same class as \(a\). Since, as is easily seen, the number of such classes is finite, it suffices to consider elements of one class. Suppose an arbitrary infinite sequence of elements \(a_k = (i^{(k)}_1, i^{(k)}_2, \ldots, i^{(k)}_n)\) of \(Q\), all belonging to the same class, is given. For definiteness we may assume that
\[
i_1^{(k)} < i_2^{(k)} < \cdots < i_n^{(k)}.
\]
Since the set \(J\) is well-ordered, we can choose an infinite subsequence of elements \(b_j = (i_1^{(j)}, i_2^{(j)}, \ldots, i_n^{(j)})\) such that
\[
i_1^{(j)} \leq i_2^{(j)} \leq \cdots.
\]
Considering this sequence, we can choose a subsequence ordering the second components, and so on. In other words, we may assume that the original sequence \(a_k\) was such that
\[
i_{r_1}^{(k)} < i_{r_2}^{(k)} < \cdots
\]
and
\[
i^{(m)}_r < i^{(n)}_r\] for all \(r = 1, 2, \ldots, n\). Clearly, \(a_k < a_m\) if and only if the number of integers \(i_r^{(k)}\) and \(i_r^{(m)}\) is not greater than the number of integers between \(i_{r_1}^{(m)}\) and \(i_{r_1+1}^{(n)}\) for all \(r = 1, 2, \ldots, n-1\). Hence, since \(n\) is finite and since the number of integers between \(i_r^{(k)}\) and \(i_{r+1}^{(k)}\) is finite for all \(r\), it follows that such an infinite sequence of elements contains a strictly increasing subsequence.

Lemma 2. The partially ordered set \((T, \triangleleft)\) is narrow.

Proof. Suppose an arbitrary infinite sequence of elements \(a_k = (q_k, r_k) = ((i_1^{(k)}, i_2^{(k)}, \ldots, i_n^{(k)}), (r_1, r_2, \ldots, r_n))\) in \(T\), where \(a_k \in P\) and \(b_k \in Q\), is given. Since \((P, \triangleleft)\) and \((Q, \triangleleft)\) are narrow, we may assume that the elements \(a_k\) from an increasing sequence in \((P, \triangleleft)\) and the \(b_k\) form an increasing sequence in \((Q, \triangleleft)\). Suppose for definiteness that \(r^{(k)}_1 < r^{(k)}_2 < \cdots < r^{(k)}_n\). It may happen that for an infinite sequence of elements \(\alpha = ((i_1, i_2, \ldots, i_m), (r_1, r_2, \ldots, r_n))\) we have \(r_n < m\). We will call the elements \((i_1, \ldots, i_{m-1}), (r_1, \ldots, r_{n+1}),\)
\((i_{m+2}, \ldots, i_m)\) the (first, second, etc.) principal parts of \(\alpha\), and \((r_1, \ldots, r_n)\) the (first, second, etc.) divisors of \(\alpha\).

From the given sequence extract an infinite subsequence of elements whose first principal parts form an increasing sequence in \((P, \triangleleft)\); from this extract an infinite subsequence whose first divisors increase; then extract an infinite subsequence whose second principal parts from an increasing sequence in \((P, \triangleleft)\), and so on. In the end we obtain an infinite increasing sequence in \((T, \triangleleft)\). The other possible cases are proved similarly.

Lemma 3. If the algebraic closure relation on \(C\) has the FBP and the partially ordered set \(S\) is narrow, then the induced algebraic closure relation on \(C \times S\) has the FBP.

Let \(R = \mathbb{Z}[x_1, x_2, \ldots]\) be the ring of polynomials in \(x_1, x_2, \ldots\) with integer coefficients and let \(M\) be the free \(R\)-module over the elements of the set \(Q\). We will call a submodule \(N\) of \(M\) a \(\Phi\)-module if