CONDITIONALLY CONVERGENT SERIES IN A UNIFORMLY SMOOTH BANACH SPACE

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The theorem of Steinitz on the form of the set of points which are sums of convergent rearrangements of a given series is extended to series \( \Sigma x_k \) in the uniformly smooth Banach space \( X \) with modulus of smoothness \( \rho(t) \), satisfying the condition \( \Sigma \rho(\|x_k\|) < \infty \).

The set
\[
\sigma(\sum x_k) = \{ s \in X; s = \sum_{k=1}^{\infty} x_{\pi(k)} \}
\]
of sums of all convergent permutations of the series \( \Sigma x_k \), whose terms are elements of the real Banach space \( X \), is called the range of sums of the series.

The set of linear functionals
\[
\Gamma(\sum x_k) = \{ f \in X^*; \sum |f(x_k)| < \infty \}
\]
is called the set of functionals of convergence of the given series; it is a linear (not necessarily closed) set.

It is easy to see that
\[
\sigma(\sum x_k) \subseteq s + \Gamma_0,
\]
where \( s \) is the sum of any rearrangement of \( \Sigma x_k \), and \( \Gamma_0 \subset X \) is the annihilator of \( \Gamma(\Sigma x_k) \). In fact, for every \( f \in \Gamma \), the series \( \Sigma f(x_k) = f(s) \) converges absolutely, and, consequently, its sum does not vary under rearrangements. Hence, for any convergent rearrangement \( \Sigma x_{\pi(k)} = s' \), we have \( f(s) = f(s') \) for all \( f \in \Gamma \), which is equivalent to (1). The convex increasing function
\[
\rho_X(t) = \sup_{u, v} \left( \frac{\|u + tv\|_2 + \|u - tv\|_2}{2} - 1 \right), \quad \text{with } \|u\| = \|v\| = 1; \ t > 0
\]
is called the modulus of smoothness of the space \( X \).

The space is called uniformly smooth if
\[
\lim_{r \to 0} \frac{\rho_X(0)}{r} = 0.
\]

In particular, the spaces $L_p(1 < p < \infty)$ are uniformly smooth, where (cf. [2]), for small $t$,
\[
\rho_{L_p}(t) = \begin{cases} 
\frac{t^p}{p} + O(t^{2p}) & \text{for } 1 < p \leq 2, \\
\frac{(p-1)t^2}{p} + O(t^3) & \text{for } p > 2.
\end{cases}
\]

In this note we prove the following.

**THEOREM.** Suppose a convergent series in a uniformly smooth space satisfies the conditions

\[ \sum x_k = s; \quad \sum \|x_k\| = \infty; \quad \sum \rho(\|x_k\|) < \infty. \]

Then its range of sums has the form

\[ \sigma = s \oplus \Gamma_0, \]

where $\Gamma_0 \subset X$ is the annihilator of the set of functionals of convergence $\Gamma(\Sigma x_k)$.

For $X = L_2$, this theorem was recently proved by Drobot (cf. [1]) under the additional assumption that $\Gamma(\Sigma x_k)$ is closed. The method of [1] is used in our note.

**LEMMA 1.** Let \( \{y_k\}_{k=1}^n \subset X; y = \sum_{k=1}^n \lambda_k y_k, \ 0 \leq \lambda_k \leq 1. \) Then there exists a vector \( \tilde{y} = \sum_{k=1}^n \delta_k y_k (\delta_k = 0 \ or \ 1) \) such that

\[ \|y - \tilde{y}\| \leq A \left[ \max_{1 \leq k \leq n} \|y_k\| \right] + \left[ \sum_{1 \leq k \leq n} \rho(\|y_k\|) \right], \]

where $A > 0$ and $0 < \gamma \leq 1$ depend only on $\rho(t)$.

**Proof.** The coefficients $\delta_k$ ($k = 1, \ldots, n$) are constructed in the following manner: we choose $\delta_1$ arbitrarily (0 or 1); we choose $\delta_2$ by the condition

\[ (\delta_2 - \lambda_2) f_{[\delta_1 - \lambda_1]}(y_2) \leq 0, \]

where $f_{[\delta_1 - \lambda_1]}y_1$ is the normalized support functional for $(\delta_1 - \lambda_1)y_1$; if $\delta_1$ is already constructed for $i = 1, \ldots, m$ and $S_m = \sum_{i=1}^m (\delta_i - \lambda_i) y_i$, we choose $\delta_{m+1}$ by the condition

\[ (\delta_{m+1} - \lambda_{m+1}) f_{S_m}(y_{m+1}) \leq 0, \]

where $f_{S_m}$ is the normalized support functional for $S_m$. The proof that \( \tilde{y} = \sum_{k=1}^n \delta_k y_k \) satisfies (2) coincides almost verbatim with the proof of similar propositions in [2] and [3] (for the case $X = L_p$, cf. [4]).

**LEMMA 2.** Let $C$ be a convex closed set in the Banach space $X$; $\Gamma$ is a subset of $X^*$; $\Gamma_0 \subset X$ its annihilator. If for every $f \in \Gamma$ and every $T > 0$, there exist $x'$ and $x''$ in $C$ such that $f(x') > T$ and $f(x'') < -T$, then $x \in C$ implies $x + \Gamma_0 \subset C$.

**Proof.** Assume that $x \in C, y \in \Gamma_0$, but $x + y \not\in C$. Then by the Hahn-Banach theorem there exists a linear functional $f$ such that

\[ f(x) < a = f(x + y) \quad \text{for all} \quad x \in C. \quad (3) \]

From (3), it is clear that $f(y) = 0$. Hence, $f \in \Gamma$, but the condition of the lemma is not fulfilled, for example, for $T = |a|$. The lemma is proved.

**LEMMA 3.** Let $\Sigma x_k$ be a convergent series in the Banach space $X$; $\Gamma_0$ the annihilator of the set of functionals of convergence $\Gamma(\Sigma x_k)$. We form two sets: $P(S)$, the set of all partial sums of terms of the series, and $Q(S)$, the set of all linear combinations of terms of the series with coefficients in $[0, 1]$ ($S$ is the set of terms of the series):