GENERAL METRIC SPACES OF LINEAR ELEMENTS
OF FIRST LACUNARITY

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We exhibit the order of the group of motions of maximally mobile spaces of linear elements.
We find all spaces of nonriemannian linear elements admitting a group of motions of maximal
order. We exhibit the maximal order of groups of motions in generalized spaces of linear elements.
Bibliography: 6 titles.

1. Riemann, in his work On the Hypotheses which Lie at the Foundation of Geometry (1868) pointed
out the possibility of introducing a metric by defining $ds$ as an arbitrary homogeneous function of first
degree in the differentials $dx^i : ds = L(x^i, dx^i), i,j = 1,2, \ldots , n$.

This approach, however, was carried out in its entirety only much later. The first such work was the
dissertation of Finsler (1918). In it Finsler gives a systematic examination of the fundamental concepts of
a metric space of linear elements and a variety of questions of the theory of curves and surfaces in it. At
the same time the theory of motions in Riemannian spaces is strongly developed.

The foundations of the theory of motions was laid by Riemann and Lie. At the end of the nineteenth
and beginning of the twentieth century appear the works of Killing, Fubini, Bianchi, and other authors on
groups of motions in Riemannian spaces. Cartan, Schouten, and Weyl advanced the idea of spaces with a
linear connection. Eisenhart and Knebelman studied affine and projective motions in these spaces.

The study of motions is conducted basically in two directions. The first is characterized by the study
of motions in prescribed Riemannian spaces and their generalizations, and the second is the construction
using the Lie group of a differential-geometric space with invariant metric or connection. The fundamental
problem in the theory of motions is the distribution of gaps and intervals of possible orders of the complete
groups of motions, the definition for the latter of the groups themselves and the lacunary spaces correspond-
ing to them. A space is said to be of lacunarity $k$ if the order of its complete group of motions belongs to
the interval of condensation having index $k$. The count starts from the interval of condensation containing
a maximal number of parameters. It is easy to see that the fundamental problem is a part of the general
problem of classification of differential-geometric spaces according to the groups of motions they admit.

One may approach the fundamental problem as follows. In the theory of motions Fubini's theorem
on the nonexistence of Riemannian spaces of nonconstant curvature whose complete group of motions is of
order $r = n(n + 1)/2 - 1$ is well known. The question arises: Does Fubini's theorem hold in the theory of
spaces with an affine connection? This question was first posed in the theory of motions and completely
solved in 1945 by I. P. Egorov [1]. In this work it was proved that the maximal order of groups of motions
$G_r$ in nonflat spaces with an affine connection is exactly $n^2$. The presence of such a gap in the orders of the
groups of motions made it possible subsequently to pose the problem of finding other gaps. Thus at first
in affine-connection spaces and later in Riemannian and other generalized spaces there arose the problem
of the distribution of the possible orders of the complete groups of motions and the study of the spaces
corresponding to them. There appeared and continues to appear at present a large number of papers of
both Soviet and foreign mathematicians on gaps and lacunary spaces.

In 1928 Douglas introduced the concept of generalized spaces of paths and in 1931 he generalized them
by constructing a theory of spaces of $k$-dimensional planar elements.

In 1946 Laptev introduced the concept of spaces of supporting elements [2], particular cases of which
are a Finsler space, a path space, and a space of $k$-dimensional planar elements. If the supporting object
in the space of supporting elements is a tensor or pseudotensor, we obtain a space of tensor supporting

elements. Particular cases of spaces of tensor supporting elements are the spaces of linear contravariant hyperplanar (covariant) elements [3].

Laptev’s spaces of supporting elements inspired a large number of studies on motions in these spaces. Laptev [4] obtained the integrability conditions for the equations of motions of spaces of tensor supporting elements in the case of a so-called truncated and symmetric connection.

2. We present several concepts and definitions that will be needed in the subsequent exposition of this article.

Suppose given a manifold \(X_n(x^1, x^2, \ldots, x^n)\). This manifold will be called the basis manifold in what follows. At each point of the basis manifold \(X_n(x)\) we shall attach (or associate) the \((n - 1)\)-dimensional space of values of the pseudovector \(y^j, j = 1, 2, \ldots, n\). The pseudovector \(y^j\) [2] will be called, in Laptev’s terminology, the supporting object. The pair consisting of the point \(x^i\) of the basis manifold \(X_n(x)\) and the supporting object \((y^j)\) will be called the linear element \((x^i, y^j)\), and the manifold so obtained will be called the space of linear elements.

All investigations here are carried out locally; therefore the spaces of linear elements under consideration are simply the topological product of the basis manifold \(X_n\) and the \((n - 1)\)-dimensional space of values of the pseudovector. The space of values of a pseudovector is also frequently called a stratum. Thus the space of linear elements is a space of ordered pairs \((x, y)\). Here \(y\) is an abbreviation for the pseudovector with coordinates \(y^j, j = 1, 2, \ldots, n\). The space of linear elements will henceforth be denoted \(X_{2n-1}(x, y)\).

In such a case the components of the metric tensor \(g_{ij}(x, y)\) now depend essentially on the coordinates of the direction \(y^i\). We remark that in the spaces \(g_{n,v}\) the tensor \(\Omega_{jk} = (g_{ik} - g_{ki})/2 \ (j, k = 1, 2, \ldots, n)\) in general is nonzero.

It is not difficult to see that if the tensors \(g_{ij}, \Omega_{jk} = 0, \Omega_{jk} = 0, (\alpha, \beta, \gamma = 1, 2, \ldots, n); j, k = 1, 2, \ldots, n,\) then the general metric spaces of linear elements \(g_{n,v}\) will reduce to ordinary Riemannian spaces \(V_{n}(x)\).

Let \(G_r\) be an \(r\)-parameter Lie group of transformations acting on the basis manifold \(X_n(x)\). We denote the finite equations of the group \(G_r\) by \(\ddot{x}^i = f^i(x^1, x^2, \ldots, x^n; a^1, a^2, \ldots, a^r)\). Then the natural extension of the group \(G_r\) to the space of linear elements \(X_{2n-1}(x, y)\) is written as follows:

\[
\tilde{x}^i = f^i(x, a), \quad \tilde{y}^j = \frac{\partial f^i(x, a)}{\partial x^\sigma} y^\sigma; \quad i, j, \sigma = 1, 2, \ldots, n.
\]

Motions (isometries) in general metric spaces of linear elements \(g_{n,v}\) are defined to be point transformations of the basis manifold \(X_n(x^1, x^2, \ldots, x^n)\) whose natural extensions to the spaces of linear elements \(X_{2n-1}(x, y)\) preserve the metric tensor \(g_{ij}(x, y)\). In order for the components of a vector field \(v^i(x)\) of the infinitesimal transformation \(\tilde{x}^i = x^i + v^i(x) t\) to define a motion in the spaces \(g_{n,v}\) it is necessary and sufficient that \(Dg_{ij} = 0\), where \(D\) is the Lie symbol for differentiation along the flow lines of the vector field \(v^i(x)\).

In the present article we study general metric spaces \(g_{n,v}\) of linear elements from the point of view of the groups of motions they admit.

We shall study only regular general metric spaces of linear elements \(g_{n,v}\), i.e., spaces admitting a unique invariant torsion-free linear connection [5]. In this connection we determine all maximally mobile general metric spaces \(g_{n,v}\). These will be the spaces \(g_{n,v}\) admitting complete groups of motion \(G_r\) of order \(n(n + 1)/2\).

3. To any general metric space of linear elements \(g_{n,v}\) one can uniquely adjoin the Finsler space \(F_{n,v}\) with metric function

\[
F(x, y) = g_{ij}(x, y)y^iy^j.
\]

We assume throughout this section that \(\det \|F_{j,k}\| \neq 0, (j, k = 1, 2, \ldots, n)\). Any motion of the general metric space of linear elements \(g_{n,v}\) is at the same time a motion of the associated Finsler space \(F_{n,v}\). Hence it follows that the group of motions \(G_r\) of the general metric space \(g_{n,v}\) is a subgroup of the group