ON RIGIDITY AND ANALYTIC INFLEXIBILITY OF SURFACES OF NEGATIVE GAUSSIAN CURVATURE UNDER A GIVEN DIRECTION OF DISPLACEMENT OF BOUNDARY POINTS

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We study the infinitesimal bendings of first and second order of regular (class $C^2$) surfaces of negative Gaussian curvature, which are bounded either a) by four asymptotic lines $g_1$, $g_2$, $g'_1$, and $g'_2$, or b) by two asymptotic lines $g_1$ and $g_2$ and a line $g$ which nowhere has asymptotic directions, under the assumption that connections are imposed on the surface permitting displacement of the points of the lines $g_1$ and $g_2$ (in the first case) and the line $g$ (in the second) in a constant direction.

Bibliography: 3 titles.

In the present work we study the infinitesimal bendings of first and second orders for simply connected regular (class $C^2$) surfaces of negative Gaussian curvature on which connections are imposed permitting displacement of points of part of the boundary in an arbitrarily prescribed constant direction.

**THEOREM 1.** Let $\Phi$ be a regular (class $C^2$) simply connected surface of negative Gaussian curvature bounded by four asymptotic lines $g_1$, $g_2$, $g'_1$, and $g'_2$ among which $g_1$ and $g'_1$ are asymptotic lines of one family and $g_2$ and $g'_2$ are asymptotic lines of a second family. If we impose connections on such a surface which permit displacement of points of the lines $g_1$ and $g_2$ in only one constant direction $c$, then the surface $\Phi$ in such a class of deformations will possess at most second-order rigidity, and consequently will be analytically inflexible.

**PROOF:** We choose as coordinate lines on the surface $\Phi$ its asymptotic lines. Let $x = x(u,v)$, where $(u,v) \in D$, be the equation of the surface $\Phi$. Let $u = u_0$ and $u = u'_0$ be the equations of the asymptotic lines $g_1$ and $g'_1$, and let $v = v_0$ and $v = v'_0$ be the equations of the lines $g_2$ and $g'_2$ respectively. In the plane of the variables $u$ and $v$ we choose a rectangular cartesian coordinate system $Ouv$. Then the region $D$ will be the rectangle $u_0 \leq u \leq u'_0$, $v_0 \leq v \leq v'_0$. A second-order deformation

$$x'(u,v,\varepsilon) = x(u,v) + 2\varepsilon z(u,v), \quad (u,v) \in D$$

will be a second-order infinitesimal bending of the surface $\Phi$ on which the connections in the hypothesis of the theorem are imposed, provided the vector-valued functions $\hat{z}(u,v)$ and $\check{z}(u,v)$ are solutions of the following system of total differential equations in the region $D$:

$$\frac{dx}{du} = 0$$

$$\frac{dx}{dv} + d\check{z} = 0$$

or the equivalent system of partial differential equations:

$$x_u + \hat{z}_u = 0$$

$$x_v + \frac{1}{2}z_v + \check{z}_v = 0$$

$$x_v + \frac{1}{2}z_v + x_u = 0$$

$$x_v + \frac{1}{2}z_v + 2z_v = 0$$

$$x_v + \frac{1}{2}z_v + x_v = 0$$

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and on the two adjacent sides \( u = u_0 \) and \( v = v_0 \) of the rectangle \( D \) these vector-valued functions satisfy the following conditions:

\[
\begin{align*}
\frac{1}{\lambda_1(u_0, v)} &= \lambda_1(u_0, v) \mathbf{c}, \quad v_0 \leq v \leq v_0', \\
\frac{1}{\lambda_2(u, v)} &= \lambda_2(u, v) \mathbf{c}, \quad u_0 \leq u \leq u_0' \quad \text{(10)}
\end{align*}
\]

where \( \lambda_1(u, v) \) and \( \lambda_2(u, v) \) are differentiable functions that characterize the magnitude of the displacement of the points of the lines \( g_1 \) and \( g_2 \) in the process of deforming the surface \( \Phi \). If we differentiate equality (10) along the line \( g_1 \) and (11) along the line \( g_2 \), we obtain

\[
\begin{align*}
\frac{1}{\lambda_1(u_0, v)} &= \lambda_1(u_0, v) \mathbf{c}, \quad v_0 \leq v \leq v_0' \quad \text{(14)} \\
\frac{1}{\lambda_2(u, v)} &= \lambda_2(u, v) \mathbf{c}, \quad u_0 \leq u \leq u_0'.
\end{align*}
\]

Hence, taking account of equalities (4) and (6), we find

\[
\begin{align*}
\lambda_1(u_0, v)(x_u(u_0, v), c) &= 0, \quad v_0 \leq v \leq v_0' \quad \text{(16)} \\
\lambda_1(u, v)(x_u(u, v), c) &= 0, \quad u_0 \leq u \leq u_0'. 
\end{align*}
\]

Depending on the form of the curves \( g_1 \) and \( g_2 \) we consider four possible cases:

1) \( g_1 \) and \( g_2 \) do not contain any planar segments lying in planes perpendicular to the vector \( \mathbf{c} \);
2) \( g_1 \) and \( g_2 \) are planar curves lying in a plane perpendicular to the vector \( \mathbf{c} \);
3) one of the curves \( g_1 \) or \( g_2 \) lies in a plane perpendicular to the vector \( \mathbf{c} \) but the other does not contain any planar segments lying in planes perpendicular to \( \mathbf{c} \);
4) at least one of the curves \( g_1 \) and \( g_2 \) contains planar segments lying in planes perpendicular to the vector \( \mathbf{c} \).

**CASE 1.** In studying the first case we find by Eqs. (16), (17), (10), and (11) that

\[
\left. \frac{1}{\lambda_0} \right|_{g_i} = \lambda_0 \mathbf{c}, \quad i = 1, 2.
\]

where \( \lambda_0 = \text{const} \).

Consider the vector-valued function

\[
\frac{1}{\lambda_0} \mathbf{z}(u, v) = \frac{1}{\lambda_0} \mathbf{z}(u, v) - \lambda_0 \mathbf{c}. 
\]

It is obvious that \( \frac{1}{\lambda_0} \mathbf{z}(u, v) \) is a bending field of the surface, and

\[
\left. \frac{1}{\lambda_0} \mathbf{z}(u, v) \right|_{g_i} = 0, \quad i = 1, 2.
\]

We denote by \( \varphi(u, v), \psi(u, v), \) and \( \chi(u, v) \) the covariant coordinates of the vector \( \frac{1}{\lambda_0} \mathbf{z}(u, v) \) in the basis \( x_u, x_v, m \), where \( m \) is the unit normal vector to the surface \( \Phi \). Then

\[
\varphi = (\frac{1}{\lambda_0}, x_u), \quad \psi = (\frac{1}{\lambda_0}, x_v), \quad \chi = (\frac{1}{\lambda_0}, m). 
\]

Since the surface \( \Phi \) is referred to the asymptotic lines, for the definition of the functions \( \varphi(u, v), \psi(u, v), \) and \( \chi(u, v) \) with regard to Eqs. (4)—(6) we obtain the following system of differential equations [3]:

\[
\begin{align*}
\frac{\partial \varphi}{\partial u} &= \Gamma_1^1 \varphi + \Gamma_1^2 \psi, \\
\frac{\partial \psi}{\partial v} &= \Gamma_2^1 \varphi + \Gamma_2^2 \psi, \\
\frac{\partial \varphi}{\partial v} + \frac{\partial \psi}{\partial u} &= 2(\Gamma_{12}^1 \varphi + \Gamma_{12}^2 \psi + M \chi),
\end{align*}
\]

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