A SUFFICIENT CONDITION FOR A POINT TO BE OF SWALLOWTAIL TYPE ON THE ENVELOPE OF A ONE-PARAMETER FAMILY OF SURFACES

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The sufficiency criterion of the title is proved. It complements the existing necessity criterion, found by Zalgaller.

We consider a family of locally simple surfaces in the space $R^3$ depending on the parameter $\varphi$:

$$r(u, v, \varphi) \in C^4, \quad (u, v) \in G, \quad a < \varphi < b,$$

where $G$ is some region in the plane of the parameters $(u, v)$. Zalgaller [Theory of Envelopes, Nauka, Moscow (1975)] has given necessary and (separately) sufficient conditions for the existence of an envelope of the family (1), as well as the presence on an envelope of a special line (edge of regression). In the same work an interesting type of singularity of the envelope is highlighted—the so-called point of swallowtail type, for which only necessary conditions are given. The present article is devoted to finding sufficiency conditions for the presence of a point of swallowtail type on the envelope of the family (1). The problem was posed to the author by Zalgaller.

We shall say that a regular curve $L$ is an edge of regression for the surface $\sigma$ if at points not on $L$ but sufficiently close to $L$ the surface $\sigma$ has a contingency in the form of a half-plane varying continuously along $L$. We call the point $M$ a point of swallowtail type for the surface $\sigma$ if $M$ is a point of regression of a curve $L$ of the surface $\sigma$ and away from $M$ the line $L$ is an edge of regression of $\sigma$, while at the point $M$ the surface has a contingency in the form of a plane.

Suppose the envelope of the family (1) consists of two portions extending up to a single curve $L$, and $L$ is an edge of regression of the surface $\sigma$ formed by the union of these two portions and the curve $L$. We shall include the curve $L$ in the envelope if to each point $M$ of the curve $L$ there corresponds by the law of adjunction a unique value of the parameter $\varphi$ and the tangent plane of the surface of family (1) corresponding to this value of the parameter $\varphi$ contains a contingency of the surface $\sigma$ at the point $M$. Analogously we include the point $M$ of swallowtail type in the envelope if by the law of adjunction there corresponds to this point a unique value of the parameter $\varphi$ and the tangent plane of the corresponding surface of the family (1) coincides with the contingency of the surface $\sigma$ at $M$.

We introduce the following notation:

\[ f = \left( r_u r_v r_{\varphi} \right), \quad g = \begin{vmatrix} f_u & f_v & f_{\varphi} \\ r_u & r_v & r_{\varphi} \\ r_{r_u} & r_{r_v} & r_{r_{\varphi}} \end{vmatrix}, \quad h = \begin{vmatrix} g_u & g_v & g_{\varphi} \\ r_u & r_v & r_{\varphi} \\ r_{r_u} & r_{r_v} & r_{r_{\varphi}} \end{vmatrix}, \]

\[ p = \frac{f_u f_v}{g_u g_v} = \frac{D(f, g)}{D(u, v)}, \quad g = \begin{vmatrix} h_u & h_v & h_{\varphi} \\ f_u & f_v & f_{\varphi} \\ g_u & g_v & g_{\varphi} \end{vmatrix} = \frac{D(h, f, g)}{D(u, v, \varphi)}, \]

\[ W = \begin{vmatrix} r_u^2 & r_u r_v & r_v^2 \\ r_u & r_v & r_{r_u} \\ r_{r_u} & r_{r_v} & r_{r_{r_u}} \end{vmatrix} = \left( r_u \times r_v \right)^2, \quad \alpha = \frac{1}{W} \begin{vmatrix} r_u r_{\varphi} & r_u r_v & r_v r_{\varphi} \\ r_u & r_v & r_{r_u} \\ r_{r_u} & r_{r_v} & r_{r_{r_u}} \end{vmatrix}, \quad \beta = \frac{1}{W} \begin{vmatrix} r_u^2 & r_u r_v & r_v^2 \\ r_u & r_v & r_{r_u} \\ r_{r_u} & r_{r_v} & r_{r_{r_u}} \end{vmatrix}. \]

It is easy to verify the following relations by direct computation.

\[ f_{\varphi} - \alpha f_u - \beta f_v = g W^{-1}, \]

\[ g_{\varphi} - \alpha g_u - \beta g_v = h W^{-1}. \]
By differentiating (2) on $\varphi$, $u$, and $v$, and forming the expression $g_\varphi - \alpha g_u - \beta g_v$, we find

$$f_{\varphi\varphi} - 2\alpha f_{u\varphi} - 2\beta f_{v\varphi} + \alpha^2 f_{uu} + 2\alpha\beta f_{uv} + \beta^2 f_{vv} = hW^{-2} - gW^{-2}(W_\varphi - \alpha W_u - \beta W_v) + f_u(\alpha_\varphi - \alpha\alpha_u - \beta\alpha_v) + f_v(\beta_\varphi - \alpha\beta_u - \beta\beta_v).$$

(4)

**THEOREM 1.** Suppose at the point $M(u_0, v_0, \varphi_0)$ ($u_0, v_0 \in G$, $a < \varphi_0 < b$) for the family of surfaces (1) the following conditions hold: $f = 0$, $g = 0$, $h = 0$, and

$$r_u \times r_v \neq 0, \quad p \neq 0, \quad q \neq 0.$$ 

(5)

Then if the region of variation of the parameters in the family (1) is restricted to some neighborhood $(u, v) \in G$, $a_0 < \varphi < b_0$ of the point $M$, the family (1) has an envelope for which $M$ is a point of swallowtail type.

**PROOF:** From the second condition of (5) it follows that $|f_u| + |f_v| \neq 0$. For definiteness assume

$$f_v \neq 0$$

at the point $M$. Then by continuity this inequality holds also in some neighborhood of the point $M$. Consequently the equation $f(u, v, \varphi) = 0$ has a unique solution near $M$ with respect to $v$ in the form of a function of class $C^3$: $v = v(u, \varphi) \in C^3$, and

$$v_u = -f_u f_v^{-1}, \quad v_\varphi = -f_\varphi f_v^{-1}.$$ 

(7)

Since $p \neq 0$, the system $f(u, v, \varphi) = 0$, $g(u, v, \varphi) = 0$ admits a solution with respect to $u$ and $v$ near $M$:

$$u = U(\varphi), \quad v = V(\varphi) \quad (\in C^3).$$ 

(8)

Moreover

$$U_\varphi = p^{-1} \frac{D(f, g)}{D(v, \varphi)}, \quad V_\varphi = p^{-1} \frac{D(f, g)}{D(\varphi, u)}.$$ 

(9)

Thus the set of points of the neighborhood of $M$ at which the equalities $f = g = 0$ hold simultaneously is some line (8). Consequently there exist points $M_1(u_1, v_1, \varphi_1)$ in this neighborhood at which $f = 0$ and $g \neq 0$ (the set of such points obviously depends on two parameters). But this means that at such points $M_1$ all the hypotheses of Theorem 6.11 of the book of Zalgaller mentioned above are met, and the surface $r_\varphi(u, \varphi) = r(u, v(u, \varphi), \varphi)$ is an envelope of the family (1) with $r_u \times r_v \neq 0$ in $M_1$ and the law of adjunction has the form

$$u, v(u, \varphi), \varphi \quad (\in C^3).$$ 

(10)

It follows from the equality $f = 0$ that the vectors $r_u$, $r_v$, $r_\varphi$ are coplanar on the envelope of the surface and since the first of Eqs. (5) holds, we have $r_\varphi = a_1 r_u + a_2 r_v$. Substituting this equality in the expression for $\alpha$ and $\beta$, we verify that $a_1 = \alpha$, $a_2 = \beta$, and

$$r_\varphi = \alpha r_u + \beta r_v.$$ 

(11)

We now consider the line (8) on which $f = g = 0$ (we denote this line by $L$). It follows from (2) and the first equation of (5) that on this line

$$f_\varphi = \alpha f_u + \beta f_v,$$

(12)

from which we find that $r_u \times r_v = 0$ on $L$. We shall show that $L$ is an edge of regression of the envelope and the point $M$ is a point of swallowtail type.

We replace $f_\varphi$ and $g_\varphi$ in (9) by their expressions (12) and (3):

$$U_\varphi = -\alpha + f_u h(pW)^{-1} \quad \text{and} \quad V_\varphi = -\beta - f_u h(pW)^{-1}.$$ 

(13)