A MATHEMATICAL MODEL OF A CLASS OF PROBLEMS FOR THE TRANSPORT OF A MIXTURE IN AIR

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An analytic solution of a class of boundary-value problems of mathematical physics describing the transport of a mixture in the atmosphere is considered. To solve these problems we apply the substitution method and the Fourier method. The solution of a boundary problem describing the process of contamination of the atmosphere by various substances is presented in the form of a series. The result obtained is useful for the solution of problems concerning the protection of the atmosphere.

The protection of the atmosphere from pollution is one of the most important contemporary problems. An active role in the solution of this problem is played by the scientists of our country. G. I. Marchuk gave a reasonably precise mathematical description of problems connected with the protection of the environment, especially the atmosphere, and also developed and described numerical methods for their solution [1].

Using analytic solutions one can determine optimal variants of various problems arising in the solution of practical problems, which is very difficult to do by numerical methods.

We consider here the analytic solution of a class of problems describing the transport of a mixture in air.

Suppose we are required to find a function \( \varphi(x, y, z, t) \) satisfying in a rectangular parallelepiped \( D\{0 < x < 1, 0 < y < q, 0 < z < m\} \) the conditions

\[
\begin{align*}
\mu \frac{\partial \varphi}{\partial x^2} + \nu \frac{\partial^2 \varphi}{\partial y^2} + \kappa \frac{\partial^2 \varphi}{\partial z^2} - a \frac{\partial \varphi}{\partial x} - \gamma (\varphi - \varphi_n) &= \sigma \frac{\partial \varphi}{\partial t} ; \\
\varphi(0, y, z, t) &= \varphi_1, \quad \varphi_y(1, y, z, t) = \varphi_y'(x, 0, z, t) = \varphi_y(x, q, z, t) = 0; \\
\varphi_x(x, y, z, t) &= \varphi_x(x, y, z, t) = 0; \quad \varphi(x, y, z, t_0) = \varphi_0,
\end{align*}
\]

where \( \varphi(x, y, z, t) \) is the reduced intensity of the mixture in air; \( \mu, \nu, \kappa \) are the coefficients of convective diffusion of the mixture dispersing in air (here these are constants, in the general case they are variables); \( a, \gamma, \sigma \) are constant parameters characterizing the transfer of the mixture; \( \varphi_n \) is the concentration of the limiting absorption of the mixture; \( \varphi_1 = \text{const.}, \varphi_0 = \text{const.}, 0 \leq t_0 \leq t < +\infty, t_0 = \text{const.} \)

The boundary-value problem (1)-(3) is equivalent to the following:

\[
\begin{align*}
\bar{\mu} \frac{\partial \varphi}{\partial x^2} - \bar{\nu} \frac{\partial \varphi}{\partial y} - \bar{\gamma} (\varphi - \varphi_n) &= \frac{\partial \varphi}{\partial t} ; \\
\bar{\varphi}(0, \bar{t}) &= \varphi_1, \quad \bar{\varphi}_x(1, \bar{t}) = 0, \quad \varphi(x, \bar{t}_0) = \varphi_0
\end{align*}
\]

where \( \bar{x}, \bar{t}, \bar{\mu}, \bar{a}, \bar{\gamma} \) are a special form of the dimensionless values introduced above. (In the sequel we omit the overbar)

We seek a solution of problem (4)-(5) in the form

\[
\varphi(x, \bar{t}) = u(x) + v(x, \bar{t}).
\]
The function \( u(x) \) is a solution of the boundary-value problem

\[
\mu u'' - au' - \gamma (u - \varphi_n) = 0, \quad u(0) = \varphi_1, \quad u'(1) = 0.
\]

This function has the form

\[
u(x) = (\varphi_1 - \varphi_n) \frac{\frac{r_1 e^{r_1 x + r_1} - r_1 e^{r_1 x + r_1}}{r_1 e^{r_1 x} - r_1 e^{r_1}}}{r_1 e^{r_1 x} - r_1 e^{r_1}} + \varphi_n,
\]

where

\[
\beta = \frac{a}{2\mu}, \quad r_1 = \beta - \sqrt{\beta^2 + \frac{2\gamma}{a}}, \quad r_2 = \beta + \sqrt{\beta^2 + \frac{2\gamma}{a}}.
\]

In order to find the function \( v(x, t) \) it is necessary to solve the boundary-value problem

\[
\mu \frac{\partial^2 v}{\partial x^2} - a \frac{\partial v}{\partial x} - \gamma v = \gamma v; \quad v(0, t) = \varphi.
\]

Applying the Fourier method [2], we seek its solution in the form

\[
v(x, t) = \Phi(x) T(t),
\]

which leads to a boundary-value problem for the eigenvalues

\[
\mu \Phi'' - a \Phi' - \lambda \Phi = 0, \quad \Phi(0) = \Phi'(1) = 0
\]

and to a solution of the equation

\[
T' + (\lambda_n + \gamma) T = 0,
\]

where \( \lambda_n \) are the eigenvalues of the boundary-value problem (12), which are determined from the equation

\[
tan \frac{\beta_n}{2\mu} = -\frac{\beta_n}{a},
\]

where

\[
\beta_n = \sqrt{4\lambda_n \mu - a^2},
\]

and the values of \( r_1 \) and \( r_2 \) are determined by the identities

\[
r_1 = \frac{(a - \beta_n)}{2\mu}, \quad r_2 = \frac{(a + \beta_n)}{2\mu}.
\]

Consequently, the eigenfunctions

\[
\Phi_n(x) = B_n e^{\frac{\beta_n}{2\mu} x} \sin \frac{\beta_n}{2\mu} x,
\]

and the functions \( T_n(t) \) are written in the form

\[
T_n(t) = C_n e^{-\lambda_n t + \gamma t}.
\]

From (17) and (18)

\[
v(x, t) = e^{\frac{a}{2\mu} x} \sum_{n=1}^{+\infty} A_n e^{-(\lambda_n + \gamma) t} \sin \frac{\beta_n}{2\mu} x.
\]