A system of differential equations is obtained for an N-link plane manipulator with rotary kinematic couples. The problem of the optimal design of manipulators with hydraulically driven links is considered. The optimal parameters of the mechanism are computed for a specific example.

The mechanical portion of a modern manipulator consists of a grasper (the working organ) and a "hand" mechanism, which moves the grasper within the boundaries of the working zone according to signals from the system control. Below we examine a plane manipulator mechanism containing N links, the kinematic scheme of which is presented in Fig. 1. We assume that the drive for the i-th link is located at link i - 1.

Each link of the scheme is related to a cartesian coordinate system $O_i x_i y_i$. The transition matrices are written in the form

$$A_i = \begin{bmatrix} \cos \varphi_{i-1} & -\sin \varphi_{i-1} & l_i \cos \varphi_{i-1} \\ \sin \varphi_{i-1} & \cos \varphi_{i-1} & l_i \sin \varphi_{i-1} \\ 0 & 0 & 1 \end{bmatrix}, \quad (i = 1, 2, \ldots, N),$$

where $\varphi_{i-1}$ are relative rotation angles of the links; $l_i$ are the lengths of the links.

If the point coordinates of a link i relative to link j are denoted by the column vector $r_{ji} = (x_{ji}, y_{ji}, 1)^t$, then the relation between coordinate systems i and i - 1 can be expressed by the transformation

$$R_{i-1i} = A_i R_{ii}.$$

For a static coordinate system related to a strut of the manipulator we obtain

$$R_{0i} = B_i R_{ii},$$

where

$$B_i = A_1 A_2 \ldots A_i = \begin{bmatrix} \cos q_i & -\sin q_i & \sum_{j=1}^{i-1} l_j \cos q_j \\ \sin q_i & \cos q_i & \sum_{j=1}^{i-1} l_j \sin q_j \\ 0 & 0 & 1 \end{bmatrix};$$

$q_i = \sum_{j=1}^{i} \varphi_{j-1} \quad$ are generalized coordinates.

The velocity acceleration of a point of a moving link i is obtained by differentiating (1) with respect to time:

$$\ddot{R}_{0i} = \frac{d}{dt} (B_i) R_{ii} = \dot{B}_i R_{ii}; \quad \dddot{R}_{0i} = \ddot{B}_i R_{ii}.$$
The kinetic and potential energy of all the links of the manipulator [3, 4] is

\[ T = \frac{1}{2} \sum_{i=1}^{N} \text{tr} (\dot{B}_i H_i \dot{B}_i), \]  

(2)

\[ \Pi = \sum_{i=1}^{N} m_i g B_i R_{ii}^*, \]  

(3)

where

\[ H_i = \int_{m_i} \left( R_{ii} R_{ii}^* \right) \, dm_i. \]  

(4)

Here the function \( H_i \) describes the inertial characteristics of the link; \( m_i \) is the mass of the link; \( R_{ii}^* \) are the coordinates of the center of mass; \( G^i = (0, g, 0) \); \( g \) is the acceleration of free fall; the symbols \( t \) and \( \text{tr} \) denote respectively the transposition and trace operations, the latter equalling the sum of the diagonal elements.

If we assume that the mass of link \( i \) is concentrated at a point with coordinates \( R_{ii}^* = (-\rho_i, 0, 1)^t \), then transforming relations (2), (3), and (4) we obtain

\[ T = \frac{1}{2} \left\{ \sum_{i=1}^{N} m_i \rho_i^2 \dot{q}_i^2 - 2 \sum_{i=1}^{N} m_i \rho_i \dot{q}_i \sum_{j=1}^{i} \dot{q}_j \cos (q_j - q_i) + \right. \]

\[ \left. + \sum_{i=1}^{N} m_i \left[ \sum_{j=1}^{i} \dot{q}_j \sin q_j \right]^2 \right\} + \sum_{i=1}^{N} m_i \left[ \sum_{j=1}^{i} \dot{q}_j \cos q_j \right]^2, \]

(5)

\[ \Pi = \sum_{i=1}^{N} m_i g B_i R_{ii}^*. \]

Using a second-order Lagrange equation, we obtain a dynamical model of an \( N \)-link manipulator with non-deforming links:

\[ m_i \rho_i \ddot{q}_i + l_i \sum_{j=1}^{N} m_j \dot{q}_j \ddot{q}_j - m_i \dot{q}_i \sum_{j=1}^{i} \dot{q}_j \cos (q_j - q_i) + \]

\[ + g \left( l_i \sum_{j=1}^{N} m_j - \rho_i m_i \right) \cos q_i = Q_i, \quad (i = 1, 2, \ldots, N), \]

(6)

where

\[ z_{ii} = \cos (q_j - q_i) \dot{q}_j - \sin (q_j - q_i) \dot{q}_i. \]

The generalized forces \( Q_i \) are determined from the conditions for the equality between the elementary workings of these forces on possible motions and the work of external forces applied to the links of the manipulator.

We examine the case in which generalized forces are formed by the hydraulic drives of the forward action (Fig. 2), and we denote by \( F_i \) the force on the coupling rod of the hydraulic cylinder of the \( i \)-th link. The variable denoting the length \( L_i \) of a line of force for \( F_i \) is determined from the relation