SOLUTION OF A PROBLEM IN THE OPTIMAL ESTIMATION OF THE STATES OF LINEAR DYNAMICAL SYSTEMS

V. A. Stoyan

A problem in the optimal estimation of a linear dynamical system is solved under the condition that the effects on the system of noise and observation error belong to a prescribed parallelepiped region.

Methods for analyzing linear dynamical systems under conditions of indeterminacy have recently undergone broad development. In particular, [1, 2] extensively study systems for which the effects of noise are restricted to an ellipsoidal region. We introduce below a method for the optimal estimation of the states of linear dynamical systems in which the domain of noise effects is a parallelepiped. The computational formulas obtained for such a case are fairly simple, do not require a large number of computations, and yield an estimate which is the best possible within the framework of the present formulation of the problem.

1. We consider the problem of constructing an estimate \( \hat{x}(t) \) of the state of the linear dynamical system from an observation

\[
\frac{dx(t)}{dt} = Ax(t) + f_1(t); \\
t \in [t_0, t_1]; \quad x(t_0) = x_0
\]

from the observation

\[
z(t) = Gx(t) + f_2(t)
\]

of it under the condition that the perturbations \( f_0, f_1(t) \) and observation errors \( f_2(t) \) belong to the region

\[
\Omega = \{(f_0, f_1, f_2)/\sup_{t \in \Omega} \max |f_i(t)| \leq \bar{f}\},
\]

where \( \bar{f} \) is a given constant and

\[
\hat{x}(t) \triangleq \arg \inf_{\hat{x}} \sup_{t_1} \| \hat{x}(t) - x(t_1) \|. \quad (4)
\]

Here \( x(t) \triangleq (x_1(t), x_2(t), ..., x_n(t))^* \), \( f_0, f_1 \) are n-dimensional vectors; \( f_2 \) is an m-dimensional vector; \( A \) and \( G \) are given matrices of dimension \( n \times n \) and \( m \times n \), respectively, and

\[
\| \hat{x} - x \| \triangleq \max_{t \in [t_0, t_1]} |\hat{x}_i - x_i|.
\]

We note that the assumption of the independence of the matrices \( A \) and \( G \) on the time coordinate \( t \) is inessential for the method of solution described below and is adopted only to simplify the computations.

It is known [1] that the problem (1)-(5), under the condition that

\[
\frac{\hat{x}(t)}{t} \triangleq \int_{t_0}^{t_1} \frac{\alpha}{t} \int_{t_0}^{t} \omega^* \alpha z(t) \, d\nu
\]

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where \( \hat{x}_i(t) \) is the \( i \)-th component of the vector \( \hat{x}(t) \), leads to the following optimal control problem:

\[
\begin{align*}
\frac{d\xi(t)}{dt} &= -A^*\xi(t) + G^*(t)u_i(t, \tau); \\
\xi(t_0) &= 0; \quad \xi(t_1) = e_t; \\
u^\text{opt}(t, \tau) &= \arg \min_{u(t, \tau)} \int_{t_0}^{t_1} \left( \sum_{i=1}^{m} \|\xi(t)\| + \sum_{i=1}^{m} |u_{i0}(t, \tau)| \right) dt,
\end{align*}
\]

(7) (8) (9)

where \( e_t \) is a unit vector and an asterisk denotes the operation of matrix transposition.

In solving problem (1)-(5) we assume that system (7) is controllable and that the rank of the matrix

\[
D = (G^*, QG^*, ..., Q^{j-1}G^*, Q^{j-1}(G^*)^T),
\]

where \( Q = \mathbb{I} + A^*\Delta t \) (\( \mathbb{I} \) is the identity matrix, \( \Delta t \) is a given constant), is equal to \( n \) for \( j \leq n \).

2. In solving problem (7)-(9) the observation interval \([t_0, t_1]\) is divided into \( IJ \) subintervals of length \( \Delta t \) each (\( I \) is an arbitrary integer and \( J \) is determined by the structure of the matrix \( D \)). The partition points, and also the values of the control vector \( u_i(t, \tau) \) and the phase vector \( \xi(t) \) of system (7) at these points is denoted by \( t_j, u(i,j), \xi(i,j), (i = 1, ..., I; \ j = 0, ..., J) \), respectively. We consider that \( u(i, J) = u(i + 1, 0), \xi(i, J) = \xi(i + 1, 0) \) for \( i = 1, ..., I - 1 \).

Supposing that \( d\xi(t)/dt = (\xi(i, j + 1) - \xi(i, j))/\Delta t \), problem (7)-(9) is transformed into the following difference scheme:

\[
\begin{align*}
\xi(i, 0) &= Q^i\xi(i, i + 1) - \Delta tG^*u(i, j); \\
\xi(1, 0) &= 0; \quad \xi(1, J) = e_t; \\
\sum_{i=1}^{I} \sum_{j=0}^{J-1} (\|\xi(i, j)\| + \|u(i, j)\|) \Delta t &\to \min_{u(i, j), \xi(i,j)}.
\end{align*}
\]

(10) (11) (12)

Here and below

\[
\|\xi\| = \sum_{k} |\xi_k|.
\]

(13)

After recurrent application of relation (10) to the set of points with fixed index \( i = [1, ..., I] \) we obtain

\[
D(u(i, 0), u(i, 1), ..., u(i, J - 1)) = (Q^j\xi(i, j) - \xi(i, 0))/\Delta t,
\]

(14)

where \( D \) is the matrix defined above.

Under an assumption of the existence of \( D^{-1} \) (in the usual sense or in a generalized sense), relations (14) permit the single-valued determination of the dependence of the control vector on the values of the vector \( \xi \) at the initial and final points of the partition intervals for \( i = 0, ..., I - 1 \). After this it is easy to also locate the trajectory of the system from Eq. (10):

\[
\xi(i, j) = Q^{j-1}(\xi(i + 1, 0) - M_j(Q^j\xi(i + 1, 0) - \xi(i, 0))),
\]

(15)

Here

\[
M_j = \sum_{k=0}^{j-1} Q^{-k}G^*D_{j-k+1}D^{-1},
\]

(16)

where \( D_j \) is determined from the \( m \times m \) identity matrix \( \mathbb{I} \) by the relation

\[
D_j = (0, 0, ..., 0, E, 0, ..., 0).
\]

(17)