On the Asymptotic Behavior of an Unstable Solution of an Inhomogeneous Stochastic Diffusion Equation

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Sufficient conditions are obtained for the normalized trajectories of an unstable solution of the one-dimensional Itô stochastic differential equation with coefficients \( a(t, x) \) and \( \sigma(t, x) \) to coincide with the normalized trajectories of the solution of the equation with coefficients \( a(x) \) and \( \sigma(x) \) as \( t \to \infty \) assuming that the coefficients \( a(t, x) \) and \( \sigma(t, x) \) have a certain average closeness to the coefficients \( a(x) \) and \( \sigma(x) \) over time as \( t \to \infty \). Bibliography: 8 titles.

Let \( \xi(t) \) be a solution of the inhomogeneous stochastic diffusion equation

\[
d\xi(t) = a(t, \xi(t)) \, dt + \sigma(t, \xi(t)) \, dw(t), \quad t > 0,
\]

where \( a(t, x) \) and \( \sigma(t, x) \) are positive real-valued functions that are continuous for \( t \) and \( x \) in \((-\infty, \infty)\) and such that for any \( N > 0 \) there exists a constant \( K_N \) such that

\[
a^2(t, x) + \sigma^2(t, x) \leq K_N(1 + x^2)
\]

uniformly for \( t \in [0, N] \); \( w(t) \) is a Wiener process defined on the probability space \((\Omega, \mathcal{F}, P)\); and \( \xi(0) \) is a given random quantity independent of \( w(t) \).

In the present work we study the weak convergence of the process \( \xi_T(t) = T^{-1/2}\xi(Tt) \) to a generalized diffusion process as \( T \to \infty \), assuming that there exist continuous functions \( a(x) > 0, \sigma(x) > 0 \) that are in a certain sense close to the coefficients \( a(t, x), \sigma(t, x) \) respectively of Eq. (1) as \( t \to \infty \).

The results we obtain complement those of the second author [2], who studied a similar question for a smaller class of equations (1) and restricted the investigation to convergence of distributions. Later [3] the second author obtained results on the weak convergence of the process \( |\xi_T(t)| \) to a Bessel diffusion process. In the homogeneous case when \( a(t, x) = a(x) \), and \( \sigma(t, x) = \sigma(x) \), for a certain class of equations there are necessary and sufficient conditions for weak convergence of \( \xi_T(t) \) to a generalized diffusion process (cf. [4]) and sufficient conditions for weak convergence of \( |\xi_T(t)| \) to a Bessel process [8].

We introduce the parameter \( 0 < T_0 \leq T \) and we use the notation

\[
\zeta_T(t) = T^{-1/2} f(\xi(tT)), \quad w_T(t) = T^{-1/2} w(tT), \quad f(x) = \int_0^x \exp \left\{ -2 \int_0^u a(v)\sigma^{-2}(v) \, dv \right\} \, du.
\]

The function inverse to \( f(x) \) is denoted \( \varphi(x) \); \( \varphi_T(x) = f'(\varphi(x\sqrt{T}))\sigma(\varphi(x\sqrt{T})) \); \( \zeta(t) \) is a solution of the stochastic diffusion equation

\[
\zeta(t) = \int_0^t \tilde{\sigma}(\zeta(s)) \, dw(s), \quad (2)
\]

\( \tilde{\sigma}(x) = \sigma_1 \) for \( x > 0 \), \( \tilde{\sigma}(x) = \sigma_2 \) for \( x < 0 \), and

\[
\beta_1(t) = \sup_{-\infty < x < \infty} \left| \frac{a(t, x) - a(x)}{\sigma(x)} \right|, \quad \beta_2(t) = \sup_{-\infty < x < \infty} \left| \frac{\sigma(t, x) - \sigma(x)}{\sigma^3 - \sigma(x)} \right|.
\]
THEOREM 1. Let \( \xi(t) \) be a solution of Eq. (1) and suppose there exist continuous functions \( a(x), \sigma(x) > 0 \) such that

1) \( \lim_{t \to \infty} \frac{1}{\sqrt{t}} \int_0^t \beta_1(s) \, ds = 0, \quad \lim_{t \to \infty} \beta_2(t) = 0, \)

\[
\int_{-\infty}^{\infty} \frac{|a(x)|}{\sigma^2(x)} \, dx < \infty; \quad 0 < \delta \leq f'(x)\sigma(x) \leq C;
\]

2) \( \frac{1}{f(x)} \int_0^x \frac{du}{f'(u)\sigma^2(u)} \to \begin{cases} \frac{1}{\sigma_1^2}, & x \to +\infty, \\ \frac{1}{\sigma_2^2}, & x \to -\infty. \end{cases} \)

If 1) \( 0 < \delta_0 \leq \sigma(x) \leq C[1 + |x|], \alpha \geq 2, \) or

2) \( 0 < \sigma(x) \leq C \) and \( \alpha \leq 2, \) then the process \( \xi_T(t) \) converges weakly to the process \( \xi(t) \) as \( T \to \infty. \)

PROOF: Let \( \alpha \geq 2. \) Since the function \( f(x) \) is twice continuously differentiable and

\[ f'(x)a(x) + \frac{1}{2} f''(x)\sigma^2(x) = 0 \]

for all \( x. \) Using Itô's formula we obtain

\[ \xi_T(t) = \xi_T(0) + \int_0^t \dot{\sigma}_T(\xi_T(s)) \, dw_T(s) + \sum_{k=1}^{3} J_k(t), \tag{3} \]

where

\[ J_1(t) = \sqrt{T} \int_0^t f'(\xi(sT)) \Delta a(sT) \, ds, \]

\[ \Delta a(t) = a(t, \xi(t)) - a(\xi(t)), \]

\[ J_2(t) = \frac{\sqrt{T}}{2} \int_0^t f''(\xi(sT)) \Delta \sigma^2(sT) \, ds, \]

\[ \Delta \sigma^2(t) = \sigma^2(t, \xi(t)) - \sigma^2(\xi(t)), \]

\[ J_3(t) = \int_0^t f'(\xi(sT)) \Delta \sigma(sT) \, dw_T(s), \]

\[ \Delta \sigma(t) = \sigma(t, \xi(t)) - \sigma(\xi(t)). \]

It is easy to see that

\[
\sup_{0 \leq t \leq N} |J_1(t)| \leq C \sqrt{T} \int_0^t \beta_1(sT) \, ds \to 0 \tag{4}
\]

as \( T \to \infty; \)

\[
M \sup_{0 \leq t \leq N} J_2(t) \leq 4 \int_0^N M(f'(\xi(sT))\Delta \sigma(sT))^2 \, ds \leq C \frac{1}{\delta_0^{2\alpha-4}} \int_0^N \beta_2^2(sT) \, ds \to 0
\]

as \( T \to \infty; \)

\[
\sup_{0 \leq t \leq N} |J_3(t)| = \sup_{0 \leq t \leq N} \frac{\sqrt{T}}{2} \left| \int_0^t f'(\xi(sT)) \left( \frac{-2a(\xi(sT))}{\sigma^2(\xi(sT))} \right) \Delta \sigma^2(\xi(sT)) \, ds \right|
\]

\[
\leq C \sqrt{T} \int_0^N \frac{|a(\xi(sT))|}{\sigma^{\alpha-1}(\xi(sT))} \left[ \frac{1}{\delta_0^{\alpha-2} \beta_2^2(sT) + 2\beta_2(sT)} \right] \, ds.
\]

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