The Integral Representation of the Exponential Product of Stochastic Semigroups

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An integral representation is obtained for the exponential product of stochastic semigroups

\[ X^t_s \otimes Z^t_s = X^t_s + \int_{s<u<t} X^t_u dV_u X^u_s + \int_{s<u_1<u_2<t} X^t_{u_2} dV_{u_2} X^u_{u_1} dV_{u_1} X^u_s + \cdots, \]

where \( V_t \) is the generating process of the semigroup \( Z^t_s \) and the integrals are understood in the sense of mean-square limits of the Riemann-Stieltjes sums. This representation is different from the traditional representation

\[ X^t_s \otimes Z^t_s = E + \int_{s<u<t} dW_u + \int_{s<u_1<u_2<t} dW_{u_2} dW_{u_1} + \cdots, \]

in which the integration extends over the process \( W_t = Y_t + V_t \) that is the generating process of the exponential product \( X^t_s \otimes Z^t_s \) and \( Y_t \) is the generator of the semigroup \( X^t_s \). Bibliography: 3 titles.

Suppose a family of \( \sigma \)-algebras \( \mathcal{F}_s^t \subset \mathcal{F}, \ 0 \leq s \leq t \leq T \), is defined on a family of probability spaces and that \( \mathcal{F}_s^t \subset \mathcal{F}_u^s \) for \([s,t] \subset [u,v]\) and for all \( s \leq t \) the \( \sigma \)-algebras \( \mathcal{F}_s^t \) and \( \mathcal{F}_u^s \) are independent. We shall denote by \( \mathcal{M} \) the class of stochastic semigroups \( X^t_s \) subordinate to the flow \( \mathcal{F}_s^t \) with right-continuous trajectories. The phase space is the ring \( \mathcal{R}_1 = \mathcal{R} + \alpha E \) constructed from an abstract complete topological ring \( \mathcal{R} \) by adjoining the identity \( E \). The topology in \( \mathcal{R}_1 \) is defined by the norm \( \| \cdot \|_1 = \| \cdot \| + |\alpha| \), where \( \| \cdot \| \) is the norm in \( \mathcal{R} \) generated by an inner product \( \langle \cdot, \cdot \rangle \).

For semigroups \( X^t_s \) and \( Z^t_s \) of the class \( \mathcal{M}^2 = \{ X^t_s : X^t_s \in \mathcal{M}, X^t_s - E \in \mathcal{R} M \| X^t_s \|_2^2 < \infty, M X^t_s = E \} \) the author [2] has proved that the Trotter product \( X^t_s \otimes Z^t_s \) exists and obtained a characterization of it in terms of the exponential product \( X^t_s \otimes Z^t_s = X^t_s \otimes Z^t_s \otimes [X, Z]^t_s \). In the present paper we obtain an integral representation for the exponential product of independent semigroups

\[ X^t_s \otimes Z^t_s = X^t_s + \int_{s<u<t} X^t_u dV_u X^u_s + \int_{s<u_1<u_2<t} X^t_{u_2} dV_{u_2} X^u_{u_1} dV_{u_1} X^u_s + \cdots. \]  

(1)

Here \( V_t \) is the generating process of the semigroup \( Z^t_s \); the integrals are understood as limits of the Riemann-Stieltjes sums, and the series converges in the sense of the norm of the space \( L_2(\Omega, \mathcal{R}_1) \). In the representation (1) the semigroups \( X^t_s \) and \( Z^t_s \) are on an equal basis, and therefore their product can also be expanded in a sum of integrals over the process \( Y_t \) that generates the semigroup \( X^t_s \):

\[ Z^t_s + \int_{s<u<t} Z^t_u dY_u Z^u_s + \int_{s<u_1<u_2<t} Z^t_{u_2} dY_{u_2} Z^u_{u_1} dY_{u_1} Z^u_s + \cdots. \]

This representation is different from the traditional one [3]

\[ X^t_s \otimes Z^t_s = E + \int_{s<u<t} dW_u + \int_{s<u_1<u_2<t} dW_{u_2} dW_{u_1} + \cdots, \]

in which the integration is carried out over the process \( W_t = Y_t + V_t \) that is the generating process of the exponential product \( X^t_s \otimes Z^t_s \).

We note that a special case of the expansion (1) was applied by A. V. Skorokhod [1] to recover a semigroup from its jump component, which is the product of the jumps whose norms are larger than \( \varepsilon > 0 \).

**Theorem.** Let \( X^t_s \) and \( Z^t_s \) be independent semigroups of the class \( \mathfrak{M}^2 \) and \( Y_t \) and \( V_t \) their generating processes. Then the integral representation (1) is valid for the semigroup \( X^t_s \otimes Z^t_s \).

**Proof:** Let \( \Lambda_n(s,t) = \{ t_k, k = 0, n \} \) be an arbitrary partition of the interval \( [s,t] \), \( s = t_0 < t_1 < \cdots < t_n = t \), and \( \lambda = \max(t_{k+1} - t_k) \). We introduce the notation \( \Delta X_{t_k} = X^t_{t_k+1} - X^t_{t_k} \), \( \Delta Z_{t_k} = Z^t_{t_k+1} - Z^t_{t_k} \), \( \Delta V_{t_k} = V_{t_k+1} - V_{t_k} \). It is easy to see that the inequality

\[
\prod_{k=0}^{n-1} (\Delta X_{t_k} + \Delta Z_{t_k} + E) = \prod_{k=0}^{n-1} (\Delta X_{t_k} + E) + \sum_{k=0}^{n-1} \prod_{i<k} (\Delta X_{t_i} + E) \Delta Z_{t_k} \prod_{i>k} (\Delta X_{t_i} + E)
\]

holds.

Iterating this equality a finite number of times, we obtain the fundamental relation

\[
\prod_{k=0}^{n-1} (\Delta X_{t_k} + \Delta Z_{t_k} + E) = X^t_s + \sum_{k=0}^{n-1} X^t_{t_k+1} \Delta Z_{t_k} X^t_s + \sum_{k<r} X^t_{t_k+1} \Delta Z_{t_r} X^t_{t_r+1} \Delta Z_{t_k} X^t_s + \cdots + X^t_{t_n} = \sum_{i=0}^{n} \mathcal{J}^n_i. \tag{2}
\]

As the author has shown [3], the left-hand side of this relation \( \prod_{k=0}^{n-1} (\Delta X_{t_k} + \Delta Z_{t_k} + E) \) converges in \( L_2(\Omega, \mathcal{F}_1) \) as \( \lambda \to 0 \) to a semigroup \( X^t_s \otimes Z^t_s \), independently of the choice of the sequence \( \Lambda_n(s,t) \). We shall now prove that each term \( \mathcal{J}^n_i \) on the right-hand side of (2) tends to the corresponding term on the right-hand side of (1) in \( L_2(\Omega, \mathcal{F}) \) and that the series composed of these limits converges. For simplicity we limit consideration to the sequence \( \mathcal{J}^n_0 \), which is typical.

We first show that the sequence \( \mathcal{J}^n_0 \) is equivalent in the sense of convergence in \( L_2(\Omega, \mathcal{F}) \)-norm to the sequence of Riemann-Stieltjes sums \( \sigma_n = \sum_{k=0}^{n-1} X^t_{t_k+1} \Delta V_{t_k} X^t_s \) for the integral \( \int_s^t X^t_u dV_u X^t_s \). We have

\[
\mathcal{J}^n_0 - \sigma_n = \sum_{k=0}^{n-1} X^t_{t_k+1} (\Delta Z_{t_k} - \Delta V_{t_k}) X^t_s. \tag{3}
\]

Since by hypothesis the semigroups \( X^t_s \) and \( Z^t_s \) are independent, the terms in (3) are pairwise orthogonal and consequently

\[
M\|\mathcal{J}^n_0 - \sigma_n\|^2 = \sum_{k=0}^{n-1} M\|X^t_{t_k+1} (\Delta Z_{t_k} - \Delta V_{t_k}) X^t_s\|^2 \leq \sum_{k=0}^{n-1} M\|X^t_{t_k+1}\|_1^2 M\|X^t_s\|_1^2 M\|\Delta Z_{t_k} - \Delta V_{t_k}\|^2. \tag{4}
\]

For fixed \( s \) (resp. \( t \)) the family of random elements \( X^t_s \) is a martingale over \( t \) (resp. \( s \)) filtering to the right (resp. left), and therefore \( \max\{M\|X^t_{t_k+1}\|_1^2, M\|X^t_s\|_1^2\} \leq M\|X^t_s\|_1^2 \).

Applying this estimate in (4), we obtain

\[
M\|\mathcal{J}^n_0 - \sigma_n\|^2 \leq (M\|X^t_s\|_1^2)^2 \sum_{k=0}^{n-1} M\|\Delta Z_{t_k} - \Delta V_{t_k}\|^2 \leq (M\|X^t_s\|_1^2)^2 M\|\sum_{k=0}^{n-1} (Z^t_{t_k+1} - E) - (V_t - V_s)\|^2 \to 0
\]