ON THE RELATIVE PRIMALITY OF $n$ AND $f(n)$

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We establish the asymptotic distribution of the set of natural numbers in the interval $[1, x]$ relatively prime to the corresponding value of an additive arithmetic function. The bibliography contains seven references.

It is well known that the probability that two natural numbers $n$ and $m$ are relatively prime is equal to $6/\pi^2$. More precisely,

$$\lim_{x \to \infty} \frac{1}{xy} \sum_{n \leq x} \sum_{m \leq y} \mathbb{1}_{(n, m) = 1} = \frac{6}{\pi^2}.$$  \hfill (1)

Here $n$ and $m$ are independent from one another. A series of authors ([1]-[4]) have considered the analogous problem in the case when $m$ is a function of $n$, i.e., the problem of the asymptotic behavior of the sum

$$T_f(x) = \sum_{\substack{n \leq x \atop (n, f(n)) = 1}} 1,$$

as $x \to \infty$. Erdős and Lorentz [3] showed that if $f(n) = f_1(n)$, where $f_1(x)$ satisfies certain smoothness conditions, then the corresponding probability is equal to $6/\pi^2$. As Hall [4] has shown, if $f(n)$ is the sum of distinct prime divisors of $n$, then

$$\sum_{\substack{n \leq x \atop (n, f(n)) = 1}} 1 = \frac{6}{\pi^2} x + O\left(\frac{x}{(\ln \ln x)^\delta (\ln \ln \ln x)^{\theta}}\right).$$ \hfill (3)

In the present paper we obtain an asymptotic series for the sum (2) under the assumption that $f(n)$ belongs to a certain class of additive functions.

**THEOREM.** Let $f(n)$ be a strongly additive function, $K(u)$ an integral polynomial of positive degree, $f(p) = K(p)$ for all primes $p$. Then for any fixed $\tau_0 > 0$

$$T_f(x) = \frac{6D}{n^2 \varphi(D)} x + \frac{x}{\ln x} \sum_{\varphi(k) \leq x} b_\alpha(x) (\ln x)^{\alpha} + O\left(\frac{x (\ln \ln x)^\delta}{(\ln x)^{\eta}}\right),$$ \hfill (4)

where

$$D = \prod_{p: p \mid (K(1), \ldots, K(p-1))} p,$$

and $\alpha$ runs in decreasing order through the positive values taken on by the real part of the expression

$$\tau(k, l) = \frac{1}{\varphi(k)} \sum_{\substack{m \leq x \atop \mu(m) = 0}} \frac{\zeta(1 - k, \zeta(s))}{\zeta(s)} e^{2\pi i \frac{m l}{k}} (k > 1, \mu(k) \neq 0, (k, l) = 1).$$ \hfill (5)
\[ b_\alpha (x) = \sum_{\beta} a_{\alpha, \beta} \cos (\beta \ln x + b_{\alpha, \beta}), \] (6)

\( \beta = \beta(\alpha) \) runs through various non-negative values \( \text{Im} \tau(k, l) \) under the condition \( \text{Re} \tau(k, l) = \alpha \), \( a_{\alpha, \beta} \) and \( b_{\alpha, \beta} \) are real constants.

We shall need to establish some auxiliary results.

**LEMMA 1.** Let

\[ K(x) = a_0 + a_1 x + \ldots + a_m x^m, \]

\( m \geq 1, a_b \) an integer, \( q > 1 \) a square-free number, \( D_q = (a_1, \ldots, a_m), \chi \) is a character of modulus \( q \).

Then

\[ \left| \sum_{x = 1}^{q-1} \chi(x) e^{2\pi i K(x)} q \right| < m^{w(q)} (q, D_q)^{1/2}, \]

where \( w(q) \) is the number of prime factors of \( q \).

In the case of prime \( q \) this inequality is obtained by the method of Mordell [5]; it is then simple to pass the general case \( q = p_1 p_2 \ldots p_s \), \( s \) prime, by using the obvious identity

\[ \sum_{x = 1}^{q-1} \chi(x) e^{2\pi i K(x)} q = \prod_{v=1}^{s} (\chi, (p_v) \sum_{x = 1}^{p_v-1} \chi_v(x) e^{2\pi i K(p_v x)/p_v}), \]

where \( P_v = q/p_v, \chi_v \) is a character of modulus \( p_v \), \( \chi_1 \chi_2 \ldots \chi_s = \chi \).

**LEMMA 2.** If \( k > 1 \) is a square-free integer, \( (l, k) = 1 \), then

\[ \sum_{p \leq x} e^{2\pi i k/p} \ln p = \tau(k, l) x + O (x \ln x/k), \]

where \( \tau(k, l) \) is defined by Eq. (5). The constant in the symbol 0 depends only on the polynomial \( K(x) \).

For the proof we note that

\[ \sum_{p \leq x} e^{2\pi i k/p} \ln p = \sum_{(a, k) = 1}^{k-1} e^{2\pi i k/a} \psi(x, k, a) + O (x \ln x). \]

According to a theorem of Page (cf., for example [6], p. 155), for \( k < e^{\alpha \sqrt{\ln x}} \)

\[ \psi(x, k, a) = \frac{x}{\phi(k)} - \frac{\chi_1(a) x^{\beta_1}}{\phi(k) \beta_1} + O (x e^{-\alpha \sqrt{\ln x}}), \]

where \( \chi_1 \) and \( \beta_1 \) are the possible exceptional character and null for modulus \( k \); therefore

\[ \sum_{p \leq x} e^{2\pi i k/p} \ln p = \tau(k, l) x - \frac{x^{\beta_1}}{\beta_1 \phi(k)} \sum_{(a, k) = 1}^{k-1} \chi_1(a) e^{2\pi i k/a} + O (k x e^{-\alpha \sqrt{\ln x}}), \]

\[ (k < e^{\alpha \sqrt{\ln x}}). \]

Bounding the sum on the right by means of the preceding lemma, we see that

\[ \sum_{p \leq x} e^{2\pi i k/p} \ln p = \tau(k, l) x + O (x \ln x/k) + O (k x e^{-\alpha \sqrt{\ln x}}) \]

(\( m \) is the degree of the polynomial \( K(u) \)). For

\[ k < e^{c_1 \sqrt{\ln x}}, c_3 = \min (c_1, c_2/2) \]

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