ENERGY INTERPRETATION OF IDENTIFIABILITY CONDITIONS IN DYNAMIC SYSTEMS

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Necessary and sufficient conditions of asymptotic identifiability of linear dynamic systems are obtained. These conditions have a straightforward energy interpretation and point to a direct relationship between convergence of the least-squares estimator and the "suppression" of the perturbation energy by the energy characteristics of the measurements of the phase state vector.

In this study, we derive necessary and sufficient conditions of asymptotic identifiability of linear dynamic systems which, unlike the standard results [7, 2-5], have a clear energy interpretations and point to a direct relationship between the convergence of the parameter estimators and the "suppression" of the perturbation energy by the energy characteristics of the measurements of the phase state vector. Identification is performed by the method of least squares (LS) [1, 6, 8].

Consider the state equation in the form

\[ x(k+1) = Ax(k) + v(k), \quad k \in \mathbb{N}, \]

where the phase state vector \( x(k) \in \mathbb{R}^n \), \( A \) is an unknown matrix, \( v(k) \) are perturbations of the system, \( \mathbb{N} \) is the set of natural numbers.

Let

\[ A^* = (a_1, \ldots, a_n), \quad X_N^* = (x(1), \ldots, x(N)), \quad V_N = (v(1), \ldots, v(N)). \]

Here \( * \) denotes the transpose. The identification error

\[ \Delta A(N) = \hat{A}(N) - A \]

is written in the form [1, 6, 8]

\[ \Delta A(N) = V_NX_N^{*+}, \]

where \( \hat{A}(N) \) is the LS estimator of the matrix \( A \), \( + \) denotes the pseudoinverse, \( \text{rank}(X_N) = n \) starting with some moment \( N \).

**LEMMA.** The estimation errors of the unknown matrix rows \( \Delta a_i(N), i = 1, \ldots, n \), and the perturbations \( v \) satisfy the "energy" equalities

\[ \| \Delta a_i(N) \|_{X_N^{*}, X_N^{*+}}^2 = \| \pi_{1=n} x_N(v_{Ni}) \|^2, \quad i = 1, n, \]

where \( \Delta A^*(N) = (\Delta a_1(N), \ldots, \Delta a_n(N)), \quad V_N^* = (v_{N1}, \ldots, v_{Nn}), \quad \pi_{1=n} X_N(\cdot) \) is the projector on the image of the matrix \( X_N \), \( \| \cdot \| \) is the Euclidean norm.

The lemma follows from the equality

\[ \pi_{1=n} x_N(v_{Ni}) = X_N^{*+} X_N^{*} v_{Ni}, \quad i = 1, n \]

and (2).
THEOREM 1. There exists a sequence \( \mu_N, N \in \mathcal{M} \) (\( \mu_N \in [0, 1] \)), such that for asymptotic identifiability of system (1) by the estimator \( \hat{A}(N) \) it is necessary and sufficient that

\[
\lim_{N \to \infty} \frac{\sum_{i=1}^{n} \| \pi_{lm} x_N (e_{N}) \|^2}{\mu_N \max_{|l| \neq 0} \sum_{k=1}^N (l^* x (k))^2 + (1 - \mu_N) \min_{|l| \neq 0} \sum_{k=1}^N (l^* x (k))^2} = 0.
\]

Proof. By our lemma,

\[
\sum_{i=1}^{n} \| \pi_{lm} x_N (v_{N}) \|^2 = \sum_{i=1}^{n} \| \Delta a_i (N) \|^2_{X_N^* X_N^*}.
\]

Denote \( \Delta a(N) = \text{column}(\Delta a_1(N), \ldots, \Delta a_n(N)) \). Then there exists a sequence \( \mu_N, N \in \mathcal{M} \) (\( \mu_N \in [0, 1] \)), such that the right-hand side of the last relationship is representable in the form

\[
\sum_{i=1}^{n} \| \Delta a_i (N) \|^2_{X_N^* X_N^*} = \Delta a^* (N) (E \otimes (X_N^* X_N)) \Delta a (N) =
\]

\[
= [\mu_N \lambda^+ (E \otimes (X_N^* X_N)) + (1 - \mu_N) \lambda^- (E \otimes (X_N^* X_N))] \| \Delta a(N) \|^2 =
\]

\[
= [\mu_N \lambda^+ (X_N^* X_N) + (1 - \mu_N) \lambda^- (X_N^* X_N)] \| \Delta A(N) \|^2.
\]

Here \( E \) is the \( n \times n \) identity matrix, \( \otimes \) is the tensor product symbol, \( \lambda^+ (\lambda^-) \) is the maximum (minimum) eigenvalue of the matrix. This equality concludes the proof.

THEOREM 2. For asymptotic identifiability of system (1) it is sufficient that

\[
\lim_{N \to \infty} \sum_{k=1}^{N} \| v(k) \|^2 / \min_{|l| \neq 0} \sum_{k=1}^{N} (l^* x(k))^2 = 0.
\]

COROLLARY 1. Assume that the energy of the phase state vector \( x(k) \) is of the same order in all directions \( l \) (\( \| l \| = 1 \)) as \( N \to \infty \), i.e.,

\[
\max_{|l| \neq 0} \sum_{k=1}^{N} (l^* x(k))^2 = \mathcal{O} \left( \min_{|l| \neq 0} \sum_{k=1}^{N} (l^* x(k))^2 \right), \quad N \to \infty.
\]

Then for asymptotic identifiability it is sufficient that

\[
\lim_{N \to \infty} \sum_{k=1}^{N} \| v(k) \|^2 / \sum_{k=1}^{N} \| x(k) \|^2 = 0.
\]

Here \( \mathcal{O}(\cdot) \) is the Landau symbol.

If the perturbations \( v \) satisfy the requirement

\[
\sum_{k=1}^{N} \| v(k) \|^2 \leq \lambda^2(N), \quad N \in \mathcal{M}
\]

(\( \lambda^2(1) > 0 \) and is nondecreasing), then Theorem 2 directly leads to a well-known result.

COROLLARY 2 [3, 4]. If the trajectory of the system is strongly wandering with exponent \( \lambda^{-2}(N) \), i.e.,

\[
\lambda^{-2}(N) \sum_{k=1}^{N} (l^* x(k))^2 \underset{N \to \infty}{\longrightarrow} \infty
\]

uniformly in \( l \) (\( \| l \| = 1 \)), then the system is asymptotically identifiable.