VI. CR-SUBMANIFOLDS IN A MANIFOLD OF ALMOST-COMPLEX STRUCTURE

N. M. Ostianu

This survey is an exposition of the results of the study of CR-submanifolds in almost-Hermitian manifolds and their subclasses. The concept of a CR-submanifold is generalized to the case of almost-complex manifolds without a metric.

INTRODUCTION

The theory of CR-submanifolds, i.e., submanifolds framed with a CR-structure has its origin in complex analysis. However, these submanifolds attracted serious attention among investigators only in the last decade [74], [38]. The connection between Cauchy–Riemann manifolds, known also as CR-manifolds, and CR-submanifolds provided the justification for applying this term to a certain class of submanifolds of Hermitian, and later, Gray manifolds.

In recent years CR-submanifolds have become an object of study among various classes of differentiable manifolds. We shall indicate below the basic areas of differential-geometric study of such submanifolds.

During the last two decades the theory of CR-manifolds has undergone a great development primarily in the area of the theory of functions, differential equations, and also differential geometry and its applications in theoretical physics [20], [21], [19], [51]. This opens wide vistas for the development of the theory of CR-structures not only on manifolds, including fiber bundles [19], but also on submanifolds.

The concept of a CR-structure arose in the study of the tangent Cauchy–Riemann equations on real submanifolds $M_n$ of the space $C^N$ [20], [52], [22].

1. We recall the definition of a CR-manifold (cf., for example, [19], [14], [21]).

A real manifold $M_n$ is called a Cauchy–Riemann manifold, or CR-manifold, if at each point $x \in M_n$ there is a distinguished subspace $\mathcal{H}^c_x(M_n)$ of the tangent space $T_x(M_n)$ depending smoothly on $x$ and each such subspace $\mathcal{H}^c_x(M_n)$ is endowed with a complex structure depending smoothly on $x$. The complex dimension of the subspace $\mathcal{H}^c_x(M_n)$ is called the CR-dimension of the manifold $M_n$. The collection of subspaces $\mathcal{H}^c_x(M_n)$ forms a complex tangent bundle $\mathcal{H}^c(M_n)$ called a CR-structure on $M_n$. The CR-structure is called involutive if $[\mathcal{H}^c, \mathcal{H}^c] \subset \mathcal{H}^c$.

On a CR-manifold there exists a real distribution $D$ and a field of endomorphisms $J : D \to D$ on it such that $J^2 = -I$. The distribution $D$ is $\text{Re}(\mathcal{H} \oplus \mathcal{H}^c)$ and $\mathcal{H} = \{X - iJX \mid X \in D\}$.


Let $M_m$ be a real submanifold in $C^N$ and $T_x(M_m)$ the tangent space at the point $x \in M_m$. We set

$$H_x(M_m) = T_x(M_m) \cap JT_x(M_m),$$

where $J$ is the operation of multiplication by the imaginary unit in $C^N_m$.

A complex structure is induced on $H_x(M_m)$. If the dimension of $H_x$ is constant on $M_m$, then $M_m$ is a CR-manifold, and the CR-structure arising naturally on it is called the CR-structure induced by the complex structure of the ambient space.

For an induced CR-structure on $M_m$ in $C^N$ the integrability conditions for the structure hold [20], [19].

If $\dim H_x = 0$, then $M_m$ is called anti-invariant, and when $\dim H_x = \dim T_x(M_m)$ it is called an invariant or holomorphic submanifold.

In the following exposition we shall suppose that these two cases are excluded.

The concept of a CR-submanifold in a manifold $M_n(g, J)$ of almost-Hermitian structure was introduced by the Rumanian geometer Besancu [28].

**DEFINITION 1.** A CR-submanifold of an almost-Hermitian manifold $M_n(g, J)$ is defined to be a real $m$-dimensional submanifold in $M_n$ on which there exists a holomorphic distribution $D$ whose orthogonal complement $D^\perp$ in $T(M_n)$ is totally real, i.e., $JD_x^\perp \subset T_x^\perp (M_n)$, where $T_x^\perp (M_n)$ is the orthogonal normal to the manifold $M_n$ at the point $x$.

This definition of CR-submanifold is the one accepted by almost all investigators. It transfers directly to the case of Hermitian manifolds.

Blair and Chen [40] prove the following theorem.

**THEOREM.** Let $M_n$ be a Hermitian manifold and $M_m$ a CR-submanifold in it. Then $M_m$ is a CR-manifold.

The authors consider this theorem to be the basic justification for using the term "CR-submanifold." The definition of a CR-submanifold in $M_n(g, J)$ admits another formulation. Indeed, when $M_m$ is a CR-submanifold, $T_x^\perp (M_m)$ (at each point $x \in M_m$) intersects the image of the tangent space $JT_x (M_m)$ in a subspace $V_x \subset T_x^\perp (M_m)$ whose dimension is the dimension of the orthogonal complement $D_x^\perp$ of the holomorphic tangent space $D_x$ and $JD_x^\perp = V_x$ (cf. also [59]).

**DEFINITION 2.** A CR-submanifold in an almost-Hermitian manifold $M_n(g, J)$ is defined to be a real submanifold $M_m$ on which there is defined a distribution $D$ of nontrivial holomorphic tangent spaces which is induced by an almost-complex structure $J$, and the orthogonal normal $T_x^\perp (M_m)$ intersects $JT_x^\perp (M_m)$ in a subspace of dimension equal to the dimension of $D_x^\perp$.

The concept of CR-submanifold thus introduced can be extended also to manifolds $M_n(J)$ not framed by the metric tensor field $g$ (cf. § 4).

By a slight systematizing of the literature devoted to the study of CR-manifolds one can point out that the largest part of the studies concerns CR-submanifolds in Kähler manifolds ([24], [40], [62], [60], [44], [46], and others) and in a complex space form ([46], [39], [44], and others). Less numerous are the studies of CR-submanifolds in Hermitian and almost-Hermitian manifolds.

There are only isolated studies of CR-submanifolds in manifolds framed with structures that can be classified as $(f \xi \eta \rho)$-structures [9], [55], except for CR-submanifolds in manifolds of contact and almost-contact structure. The article of Polyakov [17] published in the present volume is dedicated to this subject, as is [16].

Another cycle of studies is concerned with CR-submanifolds in complex-projective space [46], [69], [42], [73], [71] and on the sphere [44], [71], [61]. Papers in which CR-submanifolds are studied in quaternion spaces are still very few [23], [38].

A separate subject, but one closely connected with the theory of CR-submanifolds, is the theory of CR-products. This field [50] has attracted increasing attention recently.

In the present article we shall basically describe the results of studies of CR-submanifolds in Kähler manifolds framed with both a prescribed and an induced CR-structure, and we shall also stop to consider the possible generalizations of the concept of a CR-structure not requiring a metric to be given a priori on the ambient space.

A brief survey of the results obtained up to 1985 in the theory of CR-manifolds was published in [38].

§ 0. PRELIMINARY CONCEPTS

1. Let $M_n$ be a $C^\omega$ manifold of even real dimension $n$. When we speak of the manifold $M_n$, we shall always be considering some neighborhood $U \subset M_n$ of an arbitrary point $x_0 \in U$ and we shall denote by $x$ a variable point of this neighborhood.

Let $T^s (M_n)$ be the tangent bundle of order $s$ over $M_n$ and $T_x^s (M_n)$ the fiber of this bundle corresponding to the point $x \in M_n$.

We shall suppose that a bundle of frames of order $s : R^s_x (M_n) = \bigcup_{x \in M_n} R^s_x (M_n)$ is introduced over $M_n$, where $R^s_x$ is the set of frames at the point $x \in M_n$ with basis vectors $e_k, e_{K_1}, e_{K_2}, \ldots e_{K_s}$.