AXISYMMETRIC DEFORMATION OF FLEXIBLE ORTHOTROPIC TOROIDAL SHELLS

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Numerical solution of problems for constant-thickness toroidal shells in classical and increased-accuracy formulation is considered for various anisotropies and loads. The numerical approach is based on quasi-linearization and discrete orthogonalization methods. The results of numerical examples are presented in graphical form.

Toroidal shells are widely used in various engineering applications. The calculation of their stress—strain state is therefore of considerable interest. Toroidal shells have been studied by many authors [1, 2, 4, 5, 8-12]. Thus, a toroidal shell was considered in the classical framework in [1, 4, 8-12]. The application of the classical theory produces satisfactory results in many cases. However, when shells with considerable anisotropy of mechanical or thermophysical properties are subjected to local loads, the assumptions of classical theory require further refinement, because nonclassical factors may play an essential role in determining the stress—strain state of shells.

Various approaches have been developed for the construction of increased-accuracy models. Some increased-accuracy models are presented in [2, 5, 6]. We consider the solution of problems of constant-thickness toroidal shells in classical and increased-accuracy frameworks for various anisotropies and loads.

A complete system of differential equations in arbitrary orthogonal coordinates for one version of the increased-accuracy theory of shells is given in [6]. The original relationships assume parabolic variation of shear stresses across the shell. Setting in the equations of [6] \( A = 1, B = r = R_2 \sin \theta, \alpha_1 = s, \alpha_2 = \varphi, \) we obtain the following equations for axisymmetric deformation of toroidal shells:

\[
\begin{align*}
\frac{d(rN_1)}{ds} - \cos \theta N_2 + \frac{rQ_{13}}{R_1} &= -r\eta_1; \\
\frac{d(rM_1)}{ds} - \cos \theta M_2 - rN_1\theta_1 - rQ_{13} &= 0; \\
\left(\frac{N_1}{R_1} + \frac{N_2}{R_2}\right) r - \frac{d(rQ_{13})}{ds} &= r\eta_2; \\
\frac{d(rM_2)}{ds} - \cos \theta M_2 - rQ_{13} &= r\eta_3;
\end{align*}
\]

(1)

strain—displacement relationships

\[
\begin{align*}
e_1 &= \frac{du}{ds} + \frac{w}{R_1} + \frac{1}{2} \varphi_1^2; \\
\varphi_1 &= -\frac{du}{ds} + \frac{u}{R_1}; \\
\eta_1 &= \frac{d\varphi_1}{ds}; \\
\gamma_1 &= \frac{1}{r} \cos \theta u + \frac{w}{R_2}; \\
\eta_3 &= \frac{d\varphi_1}{ds}; \\
\gamma_2 &= \frac{1}{r} \cos \theta \varphi_1; \\
\varepsilon_3 &= \frac{1}{r} \cos \theta \varphi_1;
\end{align*}
\]

(2)
The quantities indexed $e$, $u$, and $T$ are respectively attributable to compression, surface forces, and the temperature field in the shell [6].

As noted in [6], if we take $17/1680 = 0.01$ in the expression for $D_{jj}^d$ in (4), then $M_i$ and $M_i^o$ in the elasticity relationships (3) are linearly dependent; the second and fourth equations in the system of equilibrium equations (1) are also linearly dependent; the order of the solving system of differential equations is thus reduced by 2.

Further analysis shows that the factor $17/1680$ in the original system of equations produces rapidly decaying solutions. Thus, for an orthotropic cylindrical shell of constant thickness, the characteristic equation acquires the roots $c_1 = (840c_1)^{1/2}$, where $c_1 = \lambda_4^2$, $t = h/R$, $\lambda_4 = G_{13}(1 - \nu_{12}\nu_{21})/E_1$. The correction associated with these roots is small. The solving system of equations is a "stiff system," and its numerical solution runs into certain difficulties.

Using the relationships

$$u_\rho = u \cos \theta + \omega \sin \theta; \quad u_z = u \sin \theta - \omega \cos \theta$$

and taking $\{N_1, M_2, Q_{13}^d, M_1^d, u_\rho, u_z, \theta_1, \varphi_1\}$ as the unknown functions, we obtain from Eqs. (1)-(3) after some transformations a solving system in normal form. Introducing the vector of unknowns

$$\mathbf{N} = \{N_1, M_1, Q_{13}^d, M_1^d, u_\rho, u_z, \theta_1, \varphi_1\}^T$$

we rewrite this system as

$$\frac{d\mathbf{N}}{ds} = A(s)\mathbf{N} + f(s) + \Phi(s, \mathbf{N}).$$

Here $A$ is an $n \times n$ matrix, $f$ is the column vector of the right-hand side, $\Phi(s, \mathbf{N})$ is the nonlinear part, and $n$ is the order of the system of equations.

For this system of equations, we need the boundary conditions for each case. The general form of the boundary conditions for $s = s_0$ and $s = s_N$ is

$$B_1\mathbf{N}(s_0) = b_1, \quad B_2\mathbf{N}(s_N) = b_2,$$

where $B_1, B_2$ are given $k \times n$ and $(n - k) \times n$ rectangular matrices and $b_1, b_2$ are given $k$- and $(n - k)$-vectors. From this system we obtain the equation of the classical shell theory as a particular case for $\varphi_1 = 0$ and $M_1^d = M_i/10$.

Take $R = b$, $\alpha = b/a$, $\beta = h/b$, $\lambda = E_2/E_1$, $\delta = G_{13}/E_1$ and change to dimensionless variables in (7) and (8). We obtain the boundary-value problem

$$\frac{d\mathbf{N}}{d\theta} = A(\theta)\mathbf{N} + f(\theta) + \Phi(\theta, \mathbf{N});$$

$$B_1\mathbf{N}(\theta_0) = b_1, \quad B_2\mathbf{N}(\theta_N) = b_2.$$