ON THE UNIQUENESS OF A WALSH SERIES CONVERGING ON SUBSEQUENCES OF PARTIAL SUMS

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We show that if a Walsh series whose coefficients tend towards zero is such that the subsequence of its partial sums indexed by $n_k$, where $n_k$ satisfies the condition $2^{k-1} < n_k \leq 2^k$ ($k = 0, 1, 2, \ldots$), tends everywhere, except possibly for a denumerable set, towards a bounded function $f(x)$, then this series is the Fourier series of the function $f(x)$.

It is a known fact (see [1]) that if a mild restriction is placed on the behavior of the coefficients of a Haar-system series, say, the restriction that some subsequence of partial sums tends towards zero, then all the coefficients of this series are equal to zero.

The question as to whether there is an analogous uniqueness theorem for Walsh series whose coefficients tend towards zero is an unsolved problem. Only a particular case of such a theorem has been established, namely, one involving subsequences of partial sums indexed by $2^{nk}$ (see [1]).

We consider here another particular case, one in which the corresponding indexing subsequence, while not necessarily of the form $2^{nk}$, does not grow, nevertheless, too rapidly. Namely, we prove the following theorem.

**Theorem 1.** Let the subsequence $\{n_k\}$ of positive integers satisfy the condition

$$2^{k-1} < n_k \leq 2^k \quad (k = 0, 1, 2, \ldots) \tag{1}$$

and suppose that for the partial sums $S_n(x)$ of the Walsh series

$$\sum_{n=1}^{\infty} a_n w_n(x), \quad a_n \to 0, \tag{2}$$

there is satisfied, for some number $A \geq 0$, the inequality

$$\lim_{n \to \infty} |S_{n_k}(x)| \leq A \tag{3}$$

everywhere except possibly for a denumerable set $E$. Then the series (2) is the Fourier series of the function

$$f(x) = \lim_{k \to \infty} S_{n_k}(x).$$

We shall require several definitions and lemmas.

It follows from the definition of the Walsh functions that for $2^{k-1} < n_k \leq 2^k$ ($k = 0, 1, 2, \ldots$) the sum $S_{n_k}(x)$ is constant on intervals of the form

$$\left(\frac{j-1}{2^k}, \frac{j}{2^k}\right). \tag{4}$$
For arbitrary \( j \) we call intervals of the form (4) intervals of rank \( k \) and we denote them by \( \Delta_k \). We denote the constant value assumed by the sum \( S_{nk}(x) \) on the interval \( \Delta_k \) by \( S_{nk}(\Delta_k) \).

**Lemma 1.** If for some \( n \) the following inequality holds

\[
S_n^\mu(\Delta_n) > B,
\]

for a partial sum of the series (2), then on \( \Delta_n \) we can find a set \( Q \), having the power of the continuum, such that \( S_{mn}(x) > B \) for \( x \in Q \) and for all \( m \geq n \).

This lemma follows from an analogous proposition for a Haar system (see, for example, [2]).

**Lemma 2.** Suppose that for the partial sums \( S_{nk}(x) \) of the series (2), where \( \{n_k\} \) is a sequence satisfying the condition (1), and for some nonempty perfect set \( P \), the inequality

\[
| S_{nk}(x) | < C \quad (k = 0, 1, 2, \ldots)
\]

for \( x \in P \setminus (E \cup R) \),

is satisfied, where \( E \) is a denumerable set and \( R \) is a set of binary rational points. Then a nonempty portion \( \pi \) of the set \( P \) and a number \( C \) can be found such that

\[
| S_{nk}(x) | < C \quad (k = 0, 1, 2, \ldots) \quad \text{for} \quad x \in \pi \setminus (E \cup R).
\]

The proof proceeds according to the standard scheme (see, for example, [3]).

We rely also on the following known theorems.

**Theorem A.** (Localization theorem, see [4]) For every series (2) and interval \((\alpha, \beta)\) with binary rational endpoints there exists a series

\[
\sum_{n=1}^{\infty} \alpha_n \mathbb{1}_{(\alpha, \beta)}(x),
\]

which is uniformly equiconvergent with the series (2) on \((\alpha, \beta)\) and converges uniformly to zero outside of \((\alpha, \beta)\). In addition, \( a_n \to 0 \) for \( n \to \infty \).

**Theorem B.** Suppose that for the partial sums \( S_n(x) \) of the series (2) and for some summable function \( f(x) \) the following inequality holds:

\[
\lim_{j \to \infty} S_j^\mu(x) \leq f(x) \leq \lim_{j \to \infty} S_j^\mu(x).
\]

Then the series (2) is the Fourier series of the function \( f(x) \).

Theorem B is a particular case of Theorem 3 in [5].

**Theorem C** (see [6]). If \( f \in L^2(0, 1) \), then the Fourier–Walsh series of the function \( f(x) \) converges almost everywhere to \( f(x) \).

**Theorem D** (see [7]). For the partial sums \( S_n(x) \) of the Fourier–Walsh series of the summable function \( f(x) \) and for an arbitrary interval \( \Delta_k \) of the form (4), the following equation holds:

\[
S_n^\mu(\Delta_k) = \frac{1}{\Delta_k} \int_{\Delta_k} f(t) \, dt.
\]

We proceed to the proof of Theorem 1. Let \( G \) be a set of points \( x \) such that in some interval \( \delta \), \( x \in \delta \), the inequality \(| S_{nk}(x) | \leq A \) is satisfied for arbitrary \( x \in \delta \) and for all \( k \geq k(x) \), where \( k(x) \) is the smallest of the numbers \( k \) for which \( x \in \Delta_k \subset \delta \).

Let \( P \) be the complement of \( G \). It is clear that \( P \) is closed. With the help of Lemma 1 it is easy to show that \( P \) is perfect. We wish to show that \( P \) is empty. Let us suppose the contrary; applying Lemma 2 to \( P \), we find a nonempty portion \( \pi \) of the set \( P \) and a number \( C \) such that

\[
| S_{nk}(x) | < C \quad (k = 0, 1, 2, \ldots) \quad \text{for} \quad x \in \pi \setminus (E \cup R),
\]

(5)