THE STRONG LAW OF LARGE NUMBERS
FOR HOMOGENEOUS RANDOM FIELDS

V. V. Yurinskii

We formulate, in terms of correlation functions, sufficient conditions for the applicability of the strong law of large numbers to random fields, homogeneous in the wide sense. We consider averages over spheres with center at the coordinate origin and also over rectangular parallelepipeds.

In this note we consider certain analogs of the strong law of large numbers (SLLN) for random fields homogeneous in the wide sense. We propose conditions for applicability of the SLLN expressed in terms of correlation functions.

In § 2 we explain the conditions of applicability of the SLLN for the case in which the field is averaged over spheres with center at the coordinate origin. Such averaging is rather natural for fields which are not too anisotropic. However, in a multidimensional space the selection of the region over which to average is far from being settled. Therefore, in § 3 we consider averaging over rectangular parallelepipeds. The proofs in § 2 use fairly involved estimates, the derivation of which is given in § 1 in order to obscure the discussion. The basic notation adopted is also introduced in § 1.

Conditions under which the SLLN is applicable to stationary random processes are given, for example, in [1] and [2]. The method we use below is essentially the same as in the literature cited. However, the results in § 2 refer only to fields depending on an argument of dimensionality not less than 2. This is connected with a degeneracy originating in the one-dimensional case which renders the geometrical constructions of § 1 meaningless. The theorem of § 3 is true also for stationary processes, but in this case is cruder than the results given in [1] and [2].

The technique of binary expansions, on which the method of [1] and [2] and of this note is based, has also been applied to the study of the SLLN in [3].

§ 1. We consider a scalar random field, homogeneous in the wide sense, namely, $\xi(t) \in \mathbb{C}, t \in \mathbb{R}^k$ with a zero mean ($\mathbb{E}\xi(t) = 0$), unit dispersion ($\mathbb{E} |\xi(t)|^2 = 1$), and correlation function

$$b(t) = \mathbb{E}\xi(t + s)\overline{\xi(s)} = \int_{\mathbb{R}^k} \exp \left(i(t, x)\right) F(dx).$$

(1.1)

We assume realizations of these fields to be Lebesgue measurable in $\mathbb{R}^k$ with probability 1.

In Eqs. (1.1), and throughout the sequel, we use the notation $(\cdot, \cdot)$ and $|\cdot|$ for the scalar product and corresponding norm in $\mathbb{R}^k$. A superscript in parentheses $(t^{(i)})$ denotes the coordinates of vectors belonging to $\mathbb{R}^k$ in some fixed orthonormal system of coordinates; $dt = d(t^{(i)}) \ldots d(t^{(k)})$. A dash is used to indicate the complex conjugate of a number.

We agree to use the same letter for all positive constants (thus the equations $c + c = c, c/c = c$ should cause no confusion).

V. A. Steklov Mathematics Institute, Academy of Sciences of the USSR. Translated from Matemati-
We put
\[ X(\tau) = \int_{|t|<\xi(t)} dt, \quad B(\tau) = \int_{|t|<\xi} b(t) dt, \quad \Lambda(\tau) = \int_{|t|<\xi} dt. \tag{1.2} \]

**Lemma 1.1.** For \( \tau > \sigma > 0 \)
\[ M(\tau, \sigma) = E[X(\tau)|X(0)|^2 - B(\rho)|\Re B(\rho)|d\rho, \]
where \( g(\rho, \tau, \sigma) = [(\tau + \sigma)^2 - \rho^2]^{(\tau^2 - (\tau - \sigma)^2)^{1/2}}. \)

**Lemma 1.2.** For \( \sigma < \tau < 3\sigma/2 \)
\[ E[X(\tau) + X(\tau - \tau)] \leq \frac{2\tau}{\int_0^\tau |\Re B(\rho)|d\rho, \]
where \( \gamma(\rho, \tau, \sigma) \) satisfies the inequalities
\[ |\gamma(\rho, \tau, \sigma)| \leq c(\sigma - \rho) \quad \text{for} \quad 0 \leq \rho \leq \tau - \sigma; \]
in the region \( \tau - \sigma \leq \rho \leq 2\sigma \), for \( k = 3 \) or \( k \geq 5 \),
\[ |\gamma(\rho, \tau, \sigma)| \leq c(\tau - \sigma)^2 \quad \text{for} \quad 0 < \rho < \tau - \sigma; \]
but if \( k = 2, 4 \), then for \( \tau - \sigma \leq \rho \leq 2\sigma \)
\[ |\gamma(\rho, \tau, \sigma)| \leq c(\tau - \sigma)^4 \quad \text{for} \quad 0 < \rho < \tau - \sigma; \]
finally, in the region \( 2\sigma \leq \rho \leq 2\tau \)
\[ |\gamma(\rho, \tau, \sigma)| \leq c(\tau - \sigma)^{1/3} \quad \text{for} \quad 0 < \rho < \tau - \sigma. \]

For the proof of these lemmas we require several equations. The Lebesgue measure of a \( k \)-dimensional spherical segment \( \{t: |t| < \tau, \Re(t) > \tau \cos \varphi\} \) (\( 0 < \varphi < \pi \)) is given by the expression
\[ \lambda(\tau, \varphi) = \Lambda(\tau) \int_0^\pi \sin^k \theta d\theta. \]

The volume of intersection of two \( k \)-dimensional spheres with radii \( 2\sigma \) and \( 2\tau \), with centers at a distance of \( 2\rho \) apart, is equal to
\[ \mu(\rho, \tau, \sigma) = \lambda(2\sigma, \tau - \varphi_1) + \lambda(2\tau, \varphi_2), \tag{1.3} \]
where \( \varphi_1 \) and \( \varphi_2 \) are found from the equations
\[ \cos \varphi_1 = (\tau^2 - \rho^2 - \sigma^2)/(2\rho\sigma), \quad \cos \varphi_2 = (\tau^2 + \rho^2 - \sigma^2)/(2\rho\tau), \quad 0 < \varphi_1, \varphi_2 < \pi. \]

When \( \rho \) varies from \( 0 \) to \( \tau - \sigma \), \( \varphi_1 = \varphi_2 = 0 \). When \( \rho \) varies from \( \tau - \sigma \) to \( \tau + \sigma \), \( \varphi_1 \) increases from \( 0 \) to \( \pi \); \( \varphi_2 \) increases at first to its maximum value and then decreases to \( 0 \). Keeping in mind that \( \sigma \sin \varphi_1 = \tau \sin \varphi_2 \)
we easily confirm the validity of the equation
\[ \frac{\partial}{\partial \rho} \mu(\rho, \tau, \sigma) = -2\Lambda(1)\int_0^\pi \sin^k \theta d\theta. \tag{1.4} \]

Lemmas 1.1 and 1.2 are proved by a direct calculation of the mathematical expectations of interest to us. It is obvious that
\[ M(\tau, \sigma) = \sum_{|t|<\xi} \int_{|t|<\xi} b(t) dt ds. \]

Using the change of variables \( u = t - s, v = t + s \) with the Jacobian \( 2^k \), we change over from a "double" \( 2k \)-fold integral to an "iterated" integral (with integration first with respect to the variable \( v \)) by means of the relation (1.3); then, changing to polar coordinates, we reduce this integral to the form...