ON THE TOPOLOGY OF THE EFFECTIVE SUBSETS
OF LEVEL SETS

L. E. Bazilevich

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In studying the problem of constructing selections of the distance function to subsets of a Euclidean space the concept of the effective subset of a level set arises. In this paper we establish the connectivity of effective subsets, which is important for applications.

The topology of the level sets of the distance function to subsets of a Euclidean space has been studied by many authors [3–5]. We note in particular the article of S. Ferry [4], which contains an exhaustive study of the problem of homeomorphisms of level sets with manifolds. In the author’s papers [1, 2], which are devoted to the construction of selections of the distance function to compact subsets in \( \mathbb{R}^n \), an important role is played by a special subset of a level set—the effective subset \( F(A, r) \) (for the definition cf. below).

In the present paper we establish the connectivity of the effective subset, which is essential in applications.

1. Definitions and lemmas. Let \([a, b]\) be a closed interval. We shall use the notation \([a, b[ := [a, b) \setminus \{b\}, ]a, b[ := [a, b) \setminus \{a\}, \) and \([a, b[ := [a, b) \setminus \{b\} \setminus \{a\} \). The symbols \((a, b)\) and \([a, b]\) denote respectively the line passing through the points \(a\) and \(b\) and the ray with origin at the point \(a\) passing through the point \(b\).

Let \(K(a, R) := \{x \in \mathbb{R}^n \mid d(x, a) < R\}, R > 0,\) be an open ball and \(T(a, L, \varphi)\) the \(\varphi\)-cone with vertex \(a\) over the set \(L\)—the set of points \(t \in L\) such that the angle \(\angle t a p\) is not larger than \(\varphi\) for some point \(p \in L\).

We define the complete projection of the point \(x \in \mathbb{R}^n \setminus A\) to be the set \(P(x) := \{y \in A \mid d(x, y) = d(x, A)\}\) and call a point \(p(x) \in P(x)\) a projection of the point \(x\). The interval \([x, p(x)]\) is called the projection interval with base at the point \(x\) and end at the point \(p(x)\). Given two projection intervals, either they do not intersect, or they have a only a common endpoint, or one is contained in the other. A projection interval that is not contained in any other projection interval is said to be maximal. We denote by \(K^p(x)\) the ball with diameter \(\|x, p(x)\|\). The symbols \(\partial X A, \overline{X A},\) and \(\text{Int} X A\) denote respectively the boundary, the closure, and the interior of the set \(A\) in the topological space \(X\). If \(X = \mathbb{R}^n\), the symbol \(X\) will be omitted from this notation.

Lemma 1. Let \(x \in \mathbb{R}^n \setminus A\) and \(p(x) \in P(x)\). Then for every \(y \in \partial K^p(x) \setminus \{x\}\) the distance function to \(A\) is strictly increasing as a point moves from \(y\) to \(x\) on the interval \([y, x]\).

Proof. Assume there exist points \(t_1 \) and \(t_2 \) in \([y, x]\) such that \(d(t_1, x) < d(t_2, x)\) and \(d(t_1, A) < d(t_2, A)\). Let \(M := \{z \in \mathbb{R}^n \mid d(t_1, z) < d(t_2, z)\}\). It is obvious that \(P(t_1) \subset M\). Therefore \(d(t_1, A) \geq d(t_1, M \setminus K(x, d(x, A))) \geq \frac{1}{2} \text{diam} (\partial K(x, d(x, A))) > d(t_2, p(x)) > d(t_2, A)\). Contradiction. The lemma is now proved.

Corollary. Let \(p_1(z)\) and \(p_2(z)\) be two projections of the point \(z\) such that the size \(\varphi\) of the angle \(\angle p_1(z) z p_2(z)\) is less than \(\pi\). Then for each \(r \in [R \sin \frac{\varphi}{2}, R]\), where \(R = d(z, A)\), the intersection of the \(r\)-level \(E(A, r)\) with the triangle \(\Delta p_1(z) z p_2(z)\) is a simple arc with endpoints on the intervals \([z, p_1(z)]\) and \([z, p_2(z)]\).

Proof. It follows from Lemma 1 that for every \(y \in [p_1(z), p_2(z)]\) the distance function to the compact set \(A\) is strictly increasing on the interval \([y, z]\). Since \(\max\{d(y, A) \mid y \in [p_1(z), p_2(z)]\} \leq R \sin \frac{\varphi}{2}\), the corollary is now proved.

We define the \(r\)-level of the distance function to the set \(A \) (\(r > 0\)) to be the set \(E(A, r) := \{x \in \mathbb{R}^n \mid d(x, A) = r\}\). We also introduce the notation \(E^-(A, r) := \{x \in \mathbb{R}^n \mid 0 < d(x, A) < r\}, E^+(A, r) := \{x \in \mathbb{R}^n \mid d(x, A) > r\}\). We now fix some connected component \(E^+(A, r)\) of the set \(E^+(A, r)\). We define the
effective subset of the set \( E^+(A, r) \) to be the set \( F^+(A, r) = F(A, r) \) all boundary points of \( E^+(A, r) \) that are reachable from within \( E^+(A, r) \), and we set \( E^-(A, r) := \text{Bd} \ E^+(A, r) \).

**Lemma 2.** Let \( x \in \mathbb{R}^n \setminus A, p(x) \in P(x) \), and \( y \in \{x, p(x)\} \cap E^-(A, r) \) for \( 0 < r < d(x, A) \). Then \( x \in E^+(A, r) \).

**Proof.** It follows from Lemma 1 that there exists a neighborhood \( U \) of the point \( y \) such that all points in the set \( U \cap E^+(A, r) \) can be joined to the point \( x \) by an interval contained in the set \( E^+(A, r) \). Consequently \( U \) meets only one connected component of the set \( E^+(A, r) \). But since \( y \in E^-(A, r) \), this component is \( E^+(A, r) \). The lemma is now proved.

**Lemma 3.** The set of points in \( E^-(A, r) \) that are not endpoints of maximal projection intervals is everywhere dense in the set \( E^-(A, r) \).

**Proof.** Let \( x \in E^-(A, r), p(x) \in P(x) \), and let \( y \in K(x, \varepsilon) \cap E^+(A, r) \) be arbitrary \( (\varepsilon > 0) \). Set \( z = \{y, p(y)\} \cap E(A, r) \). It is obvious that \( z \in E^-(A, r) \) and \( d(x, z) < \varepsilon \). Since \( \varepsilon \) is arbitrary, the lemma is proved.

**Corollary.** The set of points in \( E^-(A, r) \) that are not endpoints of maximal projection intervals is contained in the set \( F(A, r) \); consequently \( F(A, r) \) is everywhere dense in set \( E^-(A, r) \).

**Lemma 4.** Let \( c \in \mathbb{R}^n \setminus A, p(c) \in P(c) \), and let \( R \in [0, d(c, A)] \) be a certain number. Then there exists \( \varepsilon_0 > 0 \) sufficiently small that for any \( \varepsilon \in (0, \varepsilon_0) \) there exists \( a(\varepsilon) \in (0, \pi/2) \) such that for any \( y \in K(c, \varepsilon) \cap T(c, \{p(c)\}, a(\varepsilon)) \) and any \( p_1(y), p_2(y) \in P(y) \) the angle \( \angle p_1(y)p_2(y) \) is less than \( 2 \arcsin \frac{R}{R+\varepsilon} \).

**Proof.** We denote by \( B(\psi) \) the set of points \( x \in \mathbb{R}^n \setminus A \) for which the angle \( \angle p_1(x)p_2(x) \) is smaller than the number \( \psi \) for any \( p_1(x), p_2(x) \in P(x) \).

We set \( \varphi = \pi/2 - \arcsin \frac{R}{R+\varepsilon} \) and pass a ray \( l \) through the point \( c \) making angle \( \alpha \), \( \alpha \in (0, \pi/2 - 2\varphi) \) with the ray \([c, p(x)]\). Let \( z = l \cap K^c_{p(c)} \). For any \( y \in [c, z] \), we have \( P(y) \in T(y, \{z\}, \beta) \cap K(c, d(c, A)) \) where \( \beta \) is the size of the angle \( \angle zyp(c) \). Let \( z_0 \) be the point of intersection of \( l \) with the line passing through the point \( p(c) \) and making angle \( \alpha \) with the segment \([z, p(c)]\) inside the triangle \( \Delta zcp(c) \). Then \( B(2 \arcsin \frac{R}{R+\varepsilon}) \supset [z_0, c] \). It remains only to require that the condition \( d(z_0, c) \geq \varepsilon \) hold, i.e., \( \varepsilon + r \sin \alpha \tan \varphi \leq r \cos \alpha \). As \( \varepsilon \to 0 \) and \( \varphi \to 0 \), therefore, the last inequality will hold starting from some \( \varepsilon_0 > 0 \). The lemma is now proved.

**Theorem 1.** The effective subset \( F(A, R) \) is arcwise connected.

**Proof.** Assume that \( F(A, R) \) is not arcwise connected. Take an arbitrary \( x \in F(A, R) \) that is not the endpoint of a maximal projection interval (corollary to Lemma 3). As shown in [4], there exists a neighborhood \( V \) of the point \( x \) such that the set \( V \cap F(A, R) \) is homeomorphic to an \((n-1)\)-dimensional disk and consequently arcwise connected. Denote by \( N_1 \) the arc-component of the set \( F(A, R) \) containing \( x \). It is obvious that \( V \subset F(A, R) \subset N_1 \). Set \( N_2 := F(A, R) \setminus N_1 \). It is easy to see that \( d(N_1, N_2) = 0 \).

Fix a number \( \varepsilon > 0 \) that is very small in comparison with \( R \) and set \( S := \{z \in E^-(A, R + \varepsilon) \cap E^-(A, r) : d(z, N_1) = d(z, N_2)\} \). The set \( S \) is closed in \( E^-(A, R + \varepsilon) \) and consists of points that are endpoints of maximal projection intervals. Moreover \( S \cap E^-(A, R + \varepsilon) \neq \emptyset \) for any \( \varepsilon \in (0, \varepsilon_0) \). It is easy to see that \( d(z, E(A, R)) = d(z, E^-(A, R)) = d(z, F(A, R)) = d(z, A) - R \), and therefore \( d(z, N_1) = d(Z, N_2) = d(z, A) - R \) for \( z \in S \). Consequently the subsets \( P_i(z) \subset P(z) \) and \( P_i(z) := \{p(z) \in P(z) : \{z, p(z)\} \cap N_i \neq \emptyset\} \), \( i = 1, 2 \), are nonempty. It is obvious that \( X_i(z) \in N_i \), where \( X_i(z) := \{z, p(z) \cap E(A, R) \} \in P_i(z) \). Consequently the subsets \( P_i(z) \subset P(z) \) and \( P_i(z) := \{p(z) \in P(z) : \{z, p(z)\} \cap N_i \neq \emptyset\} \), \( i = 1, 2 \), are nonempty. It is easy to see that \( \min P_i(z) < 2d(z, A) \sin \varphi = 4d(z, A)R\sqrt{2R+\varepsilon} < 4R\sqrt{2R+\varepsilon} := k \) (\( \varphi = 2 \arcsin \frac{R}{R+\varepsilon} \)). Let \( p_i^*(z) \in P_i(z) \) be points such that \( P_i(z) \in \text{Cl}(p_i^*(z), k) \). Without loss of generality we shall assume that the point \( z \) coincides with the origin and the ray \([z, p_i^*(z)]\) points in the direction of the positive \( OX_\alpha \)-axis. As follows from the fact that a distance function is continuous, for an arbitrary