DECOMPOSITION OF POLYNOMIAL MATRICES INTO FACTORS WITH PRESCRIBED CANONICAL DIAGONAL FORMS

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We give a test for decomposability of a polynomial matrix over an arbitrary infinite field into factors with prescribed canonical diagonal forms whose product is the canonical diagonal form of the given matrix. We exhibit a method of actually constructing such decompositions of polynomial matrices.

Let $P$ be an infinite field and $P_n$ the ring of $n \times n$ matrices over $P$; let $A(x) = A_0 x^m + A_1 x^{m-1} + \cdots + A_m$ be a nonsingular polynomial matrix over the field $P$, i.e., $A_i \in P_n$, $i = 0, 1, \ldots, m$, $\det A(x) \neq 0$. We shall denote by $D^A(x)$ the canonical diagonal form of the matrix $A(x)$, i.e.,

$$D^A(x) = U(x)A(x)V(x) = \text{diag} (\varepsilon_1(x), \ldots, \varepsilon_n(x)),$$

$$\varepsilon_i(x) | \varepsilon_{i+1}(x), \quad i = 1, \ldots, n-1, \quad U(x), V(x) \in GL_n(P[x]).$$

In this paper we exhibit a method of constructing factorizations of polynomial matrices of the form $A(x) = B(x)C(x)$, where $B(x)$ is a monic polynomial matrix, i.e., $B(x) = E x^s + B_1 x^{s-1} + \cdots + B_s$, where $E$ is the identity $n \times n$ matrix and the canonical diagonal form of the matrix $A(x)$ is the product of the canonical diagonal forms of the factors $B(x)$ and $C(x)$: $D^A(x) = D^B(x)D^C(x)$. If $P$ is an algebraically closed field of characteristic zero, factorizations of this type for polynomial matrices were studied in [2].

Let the canonical diagonal form $D^A(x)$ of the matrix $A(x)$ be representable as a product

$$D^A(x) = \Phi(x)\Psi(x),$$

(1)

where $\Phi(x) = \text{diag} (\varphi_1(x), \ldots, \varphi_n(x))$, $\varphi_i(x) | \varphi_{i+1}(x)$, $\deg \det \Phi(x) = n s$, $\Psi(x) = \text{diag} (\psi_1(x), \ldots, \psi_n(x))$, $\psi_i(x) | \psi_{i+1}(x)$, $i = 1, \ldots, n-1$. On the basis of the results of [3] there exist matrices $Q \in GL_n(P)$ and $R(x) \in GL_n(P[x])$ such that

$$F(x) = QA(x)R(x) = \begin{bmatrix}
\varepsilon_1(x) & 0 & \cdots & 0 \\
f_{12}(x) & \varepsilon_2(x) & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
f_{n1}(x) & f_{n2}(x) & \cdots & \varepsilon_n(x)
\end{bmatrix},$$

(2)

where $\varepsilon_j(x) | f_{ij}(x)$, $\deg f_{ij}(x) < \deg \varepsilon_i(x)$ if $\deg \varepsilon_i(x) > 0$, and $f_{ij}(x) \equiv 0$, if $\deg \varepsilon_i(x) = 0$, $i, j = 1, \ldots, n$, $j < i$. We write the matrix $F(x)$ as follows:

$$F(x) = \begin{bmatrix}
\varphi_1(x) & 0 & \cdots & 0 \\
g_{21}(x) & \varphi_2(x) & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
g_{n1}(x) & g_{n2}(x) & \cdots & \varphi_n(x)
\end{bmatrix} \begin{bmatrix}
\psi_1(x) \\
\psi_2(x) \\
\cdots \\
\psi_n(x)
\end{bmatrix} = G(x)\Psi(x).$$

(3)

It is obvious that the matrix $G(x)$ can be brought into the form (2) by right elementary operations, i.e., for some matrix $R_1(x) \in GL_n(P[x])$

$$H(x) = G(x)R_1(x) = \begin{bmatrix}
\varphi_1(x) & 0 & \cdots & 0 \\
h_{21}(x) & \varphi_2(x) & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
h_{n1}(x) & h_{n2}(x) & \cdots & \varphi_n(x)
\end{bmatrix},$$

where $\varphi_j(x)|h_{ij}(x)$, $\deg h_{ij}(x) < \deg \varphi_i(x)$ if $\deg \varphi_i(x) > 0$ and $h_{ij}(x) \equiv 0$ if $\deg \varphi_i(x) = 0$, $i, j = 1, \ldots, n - 1$, $j < i$. Then

$$F(x) = G(x)R_1(x)R_1^{-1}(x)\Psi(x) = H(x)K(x).$$

(4)

We write the matrix $H(x)$ as a matrix polynomial

$$H(x) = H_0x^i + H_1x^{i-1} + \cdots + H_i, \quad H_i \in P_n.$$

Theorem. A polynomial matrix $A(x)$ is representable as a product

$$A(x) = B(x)C(x),$$

(5)

where $B(x) = Ex^s + B_1x^{s-1} + \cdots + B$ and $D^B(x) = \Phi(x)$, $D^C(x) = \Psi(x)$ if and only if the rank of the matrix

$$M = \begin{bmatrix}
H_0 & 0 & \cdots & 0 \\
H_1 & H_0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
H_{s-1} & H_{s-2} & \cdots & H_0 \\
\vdots & \vdots & \ddots & \vdots \\
H_{i-1} & H_{i-2} & \cdots & H_{i-s}
\end{bmatrix}$$

is $ns$.

Proof. Sufficiency. Suppose the main diagonal of the matrix $H(x)$ in relation (4) has the form

$$\Phi(x) = \text{diag}(1, \ldots, 1, \varphi_{k+1}(x), \varphi_{k+2}(x), \ldots, \varphi_n(x)),$$

where $\deg \varphi_i(x) > 0$, $i = k + 1, k + 2, \ldots, n$ and rank $M = ns$. Then, using the results of [4], it is not difficult to show that the matrix $H(x)$ is right-equivalent to a monic matrix of degree $s$. This means that there exists a matrix $T(x) \in GL_n(P[x])$ such that

$$H(x)T(x) = L(x) = Ex^s + L_1x^{s-1} + \cdots + L_s,$$

(6)

and $\deg T(x) \leq s$, i.e., $T(x) = T_0x^s + T_1x^{s-1} + \cdots + T_s$ and $T_0 = \text{diag}(1, \ldots, 1, 0, \ldots, 0)$. In Eq. (6), equating coefficients of like powers of $x$, we determine that $T_1, T_2, \ldots, T_s$ can be found as the unique solution of the linear matrix equation

$$M \cdot \begin{bmatrix}
X_1 \\
X_2 \\
\vdots \\
X_{s-1} \\
X_s
\end{bmatrix} = \begin{bmatrix}
-H_1T_0 \\
-H_2T_0 \\
\vdots \\
-H_{i-1}T_0 \\
E - H_iT_0
\end{bmatrix},$$

where $X_i$ are unknown $n \times n$ matrices, $i = 1, \ldots, s$. Then obviously the monic matrix $L(X) = H(x)T(x)$ is a left divisor of the matrix $F(x) = QA(x)R(x)$, and the matrix $B(x) = Q^{-1}L(x)Q$ is a left divisor of the matrix $A(x)$, i.e., $A(x) = B(x)C(x)$. The sufficiency is now proved.

Necessity. Suppose the factorization (5) holds for the polynomial matrix $A(x)$. Then on the basis of [3] there exist matrices $Q_1 \in GL_n(P)$ and $R_2(x), R_3(x) \in GL_n(P[x])$ such that the matrices $Q_1A(x)R_2(x) = F_1(x)$ and $Q_1B(x)R_3(x) = H_1(x)$ have the triangular form (2) with principal diagonals $\text{diag}(\varepsilon_1(x), \ldots, \varepsilon_n(x))$ and $\Phi(x) = \text{diag}(\varphi_1(x), \ldots, \varphi_n(x))$ respectively. Then from (5) we obtain

$$Q_1A(x)R_2(x) = Q_1B(x)R_3(x)R_3^{-1}(x)C(x)R_2(x),$$

or

$$F_1(x) = H_1(x)K_1(x).$$

(7)