
UPPER BOUNDS OF TOPOLOGIES

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The topology of a space \((X, \tau)\) homeomorphic to a non-\(\sigma\)-compact separable Borel set is equal to the upper bound of two topologies of the Hilbert cube. In particular, \((X, \tau)\) condenses to a compact space. The topology of a complete zero-dimensional metric space is the upper bound of two compact topologies. In particular, it dominates a compact Hausdorff topology.

In this note we prove theorems on the representation in certain cases of a topology as the upper bound of topologies of standard spaces or compact spaces. In particular, the possibility of condensing to compact spaces follows from these theorems. Therefore, the problems of this paper have a direct relationship to an old problem of Aleksandrov: Which spaces condense to compact spaces? The upper bound of compact topologies was also studied in [1].

Let us recall that by a condensation \(f: X \to Y\) of a topological space \((X, \tau)\) to \((Y, \sigma)\) we mean a one-to-one continuous mapping such that \(f(X) = Y\). If we define the topology \(\tau_i = f^{-1}(\sigma)\), where \(f^{-1}(\sigma) = \{f^{-1}(V): V \subseteq \sigma\}\), then it is clear that \(f: (X, \tau_i) \to (Y, \sigma)\) is a homeomorphism and that \(\tau \geq \tau_i\). Conversely, if \(\tau \geq \tau_i\), then \(i: (X, \tau) \to (X, \tau_i)\) is a compact space (\(i = 1, 2\)), is equivalent to the existence of compact spaces \((K_i, \sigma_i)\) and condensations \(f_i: X \to K_i (i = 1, 2)\) such that \(\tau = \sup \{f_i^{-1}(\sigma_i)\} = \sup \{f_1^{-1}(\sigma_1), f_2^{-1}(\sigma_2)\}\). By a (separable) Borel set we mean a Borel set of a complete (separable) metric space. In what follows we use the following standard notation: \(I\) is the closed interval \([0, 1]\); \(R\) is the set of real numbers; \(P\) is the set of irrational numbers; \(B(\lambda)\) is a Baire space of weight \(\lambda\), i.e., a countable power of a discrete space of weight \(\lambda\). A product of topological spaces is always treated with the Tikhonov topology. For any \(a = \{a_i\}, a \subseteq I^\infty\), we define \(I^\infty(a) = \{x = \{x_i\} \subseteq I^\infty\} \) such that \(x_i = a_i \) for only a finite number of indices \(j\). By an imbedding \(f: X \to Y\) we mean a mapping such that \(f: X \to f(X)\) is a homeomorphism.

THEOREM 1. Let \((X, \tau)\) be a topological space homeomorphic to a non-\(\sigma\)-compact separable Borel set. Then \(\tau = \sup \{\tau_i, \tau_j\}\), where \((X, \tau_j) (i = 1, 2)\) is homeomorphic to \(I^\infty\). In particular, \((X, \tau)\) condenses to \(I^\infty\).

Proof. We choose open sets \(\{V_i\} (i = 1, 2, 3, 4)\), whose closures are pairwise disjoint and which are not \(\sigma\)-compact. We shall construct condensations \(f\) and \(g\) mapping \(X\) into the product \(I \times I^{\infty} \times I^{\infty}\), homeomorphic to \(I^{\infty}\), such that:

I) The restriction \(f|(X \setminus (V_2 \cup V_4))\) is a homeomorphism;
II) the restriction \(g|(X \setminus (V_1 \cup V_3))\) is a homeomorphism;
III) \(fV_1 \cap fV_4 = \phi\) and \(fV_2 \cap fV_3 = \phi\);
IV) \(gV_2 \cap gV_4 = \phi\) and \(gV_1 \cap gV_3 = \phi\).

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Let us construct \( f \). Let \( \psi_0 \colon X \to I \) be a continuous mapping such that \( \psi_0 (V_1 \cup V_2) = 0 \) and \( \psi_0 (V_2 \cap \overline{V}_3) = 1 \). We set \( F_1 = \psi_0^{-1} [0, 1/2] \) and \( F_2 = \psi_0^{-1} (1/2, 1] \). Let \( \psi_i \colon (X \setminus (V_3 \cup V_4)) \cap F_2 = \psi_i^{-1} [0, 1/2] \) and \( \psi_i (V_2 \cap F_2) = 1 \). We set \( F_2 = \psi_i^{-1} [0, 1/2] \) and \( F_2 = \psi_i^{-1} (1/2, 1) \). Next, we shall construct \( \psi_i \colon (X \setminus (V_3 \cup V_4)) \cap F_2 = \psi_i^{-1} [0, 1/2] \) and \( \psi_i (V_2 \cap F_2) = 1 \). The constructions of \( f_1 \) and \( f_2 \) are similar, but the image of \( f_2 \) must be a more complicated set. Therefore, we shall construct \( f_2 \).

The set \( V_1 \cap F_2 \) is not \( \sigma \)-compact, and so there exists a closed set \( \Phi \subseteq V_1 \) that is not \( \sigma \)-compact. Then \( \Phi \) is a non-\( \sigma \)-compact Borel set and by [2] contains a closed set \( T \) homeomorphic to \( P \). We represent \( T \) as \( T = \bigcup \{ T_i \colon i \in \mathbb{N} \} \), where each \( T_i \) is closed and homeomorphic to \( P \), and \( \{ T_i \} \) is discrete. We define \( \psi_i \colon (F_2 \setminus V_1) \setminus T \to \mathbb{N} \) by \( \psi_i ((F_2 \setminus V_1) \setminus T_1) = 1 \) and \( \psi_i (T_i) = i, i > 1 \), and we let \( \psi_i^* \colon F_2 \to R \) be a continuous extension of \( \psi_i \). We set \( \Phi_1 = (\psi_i^*)^{-1} (\mathbb{N}, 3, 2] \) and \( \Phi_i = (\psi_i^*)^{-1} (i - 1/2, 2, i + 1/2) \).

Then \( \{ \Phi_i \} \) is a locally finite closed covering of \( F_2 \), and \( (\Phi_i \setminus \bigcup \{ \Phi_j \colon i \neq j \}) \subseteq T_i \) for any \( i \). We choose \( a_i = \{ a_i^j \} \subseteq I^i_0 \setminus \Phi_i \) such that \( \Phi_i \setminus \bigcup \{ \Phi_j \colon i \neq j \} \subseteq T_i \) for any \( i \). We choose \( \sigma \) such that \( (\Phi_i \setminus \bigcup \{ \Phi_j \colon i \neq j \}) \subseteq T_i \) for any \( i \).

We construct \( \psi_0 = \psi_0 | \Phi_0 \). Suppose that \( \psi_0, \ldots, \psi_m \) have been constructed. We construct \( \psi_{m+1} \) as follows. \( \psi_m \left( \bigcup \{ \Phi_i \} \right) \) is a Borel set, since \( \psi_m \) is injective [3]. Then \( C_0 = \{ (1/2, 1) \times I^m_0 \setminus \bigcup \{ I^m_0 (a_i) : i = k, \ldots \} \} \setminus \psi_m (\bigcup \{ \Phi_i \}) \) is also a Borel set. Since \( T_{m+1} \) is homeomorphic to \( P \), by virtue of [3] there exists a closed subset \( D_{m+1} \subseteq T_{m+1} \) and a condensation \( \eta_{m+1} : D_{m+1} \to C_0 \). We write \( [1/2, 1) \times I^m_0 \setminus \bigcup \{ I^m_0 (a_i) : i = m - 1, \ldots \} \} \setminus \psi_m (\bigcup \{ \Phi_i \}) \) as \( [1/2, 1) \times I^m_0 \setminus \bigcup \{ I^m_0 (a_i) : i = m - 1, \ldots \} \} \setminus \psi_m (\bigcup \{ \Phi_i \}) \) as follows:

\[
\begin{align*}
\pi_1^m (x), & \quad \text{if } x \in \bigcup \{ \Phi_i \}, \\
\pi_1^m (x), & \quad \text{if } x \in D_{m+1}, \\
\pi_1^m (x), & \quad \text{if } x \in M_j,
\end{align*}
\]

and we let \( \psi_i^* : \bigcup \{ \Phi_i \} \to I_j^2 \) be a continuous extension of \( \pi_j^2 \). We fix an imbedding \( \Omega_j : M_j \to \prod \{ I_i^2 \} \). We define \( \psi_i : \bigcup \{ \Phi_i \} \to I_0 \) as follows:

\[
\begin{align*}
\pi_1^m (x), & \quad \text{if } x \in \bigcup \{ \Phi_i \}, \\
\pi_1^m (x), & \quad \text{if } x \in D_{m+1}, \\
\pi_1^m (x), & \quad \text{if } x \in M_j,
\end{align*}
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