THE EXTREME POINTS OF THE UNIT BALL OF CERTAIN SPACES OF OPERATORS

I. I. Tseitlin

In this note we discuss the set of extreme points of the unit ball of certain spaces of mappings. We prove that a mapping \( T : E \to F' \) is an extreme point of the unit ball of the space \( I(E, F') \) of integral mappings, if and only if it has the form

\[
T x = \langle x, a_0 \rangle b_0, \quad \text{where} \quad a_0 \in \text{ext } S(E) \quad \text{and} \quad b_0 \in \text{ext } S(F').
\]

Holub [1] has proved that the extreme points of the unit ball of the space \( N(H) \) of nuclear mappings of a Hilbert space \( H \) are the one-dimensional mappings of norm 1, and only these. In the present note we discuss the set of extreme points of the unit ball of certain standard spaces of mappings (in arbitrary Banach spaces).

Everywhere in what follows \( E \) and \( F \) will stand for Banach spaces, and \( E' \) will stand for the dual space of \( E \). We denote by \( S(E) \) the unit ball of \( E \), and by \( \text{ext } S(F) \) the set of all the extreme points of \( S(E) \).

We recall the main definitions and the facts we require.

**Definition 1** [2]. A mapping \( T : E \to F \) is said to be nuclear if it can be represented in the form

\[
T x = \sum_{i=1}^{\infty} \langle x, a_i \rangle y_i \quad \text{for every} \quad x \in E, \quad \text{where} \quad a_i \in E', \quad y_i \in F (i = 1, 2, \ldots), \quad \text{and} \quad \sum_{i=1}^{\infty} \| a_i \| \| y_i \| < \infty.
\]

The number

\[
\nu(T) = \inf \sum_{i=1}^{\infty} |a_i| \| y_i \|.
\]
where the greatest lower bound is taken over all these representations of the mapping \( T \), is called the nuclear norm of \( T \). We denote by \( \mathcal{N}(E, F) \) the Banach space of all nuclear mappings of \( E \) into \( F \) with the norm \( \nu \).

**Definition 2 (Pietsch [2]).** A mapping \( T : E \to F \) is said to be 1-integral if it can be represented as the composition \( T = QIP \):

\[
E \xrightarrow{P} L_{\infty}(U^0, \mu) \xrightarrow{\mu} L_1(U^0, \mu) \xrightarrow{Q} F,
\]

where \( U_0 \) is the unit ball of the space \( E^0 \), \( \mu \) is a positive Radon measure on \( U^0 \). \( P \subseteq L(E, L_{\infty}(U^0, \mu)), \ Q \subseteq L(L_1(U^0, \mu), F), \ |P| \leq 1, \ |Q| \leq 1 \), and \( I \) is the identity mapping. The number

\[
\sigma_1(T) = \inf \mu(U^0)
\]

is called the 1-integral norm of \( T \). [The greatest lower bound is taken over all representations of \( T \) of the form (1).] We denote by \( \mathcal{I}_1(E, F) \) the Banach space of all 1-integral mappings of \( E \) into \( F \), with the norm \( \sigma_1 \).

Note that Definition 2 does not guarantee the existence of a representation (1) with a measure \( \mu \) such that

\[
\sigma_1(T) = \mu(U^0).
\]

This gives rise to serious difficulties in a study of the extreme points of \( \mathcal{I}_1(E, F) \). Therefore, we consider a class of mappings, introduced by Grothendieck, that is more convenient in this respect.

**Definition 3 [3].** A mapping \( T : E \to F \) is said to be integral with an integral norm \( \sigma(T) = 1 \), if it can be represented as the composition \( T = QIP \):

\[
E \xrightarrow{P} L_{\infty}(U^0, \mu) \xrightarrow{I} L_1(U^0, \mu) \xrightarrow{Q} F,
\]

where \( |P| \leq 1, |Q| \leq 1 \), \( I \) is the identity mapping, and \( \mu(U^0) = 1 \). We denote by \( \mathcal{I}(E, F) \) the Banach space of all integral mappings of \( E \) into \( F \).

It can be proved that \( \mathcal{I}_1(E, F) \subset \mathcal{I}(E, F) \).

**Proposition [3].** An integral mapping \( T : E \to F' \) with a norm \( \sigma(T) = 1 \) can be represented as the composition \( T = QIP \):

\[
E \xrightarrow{P} L_{\infty}(U^0, \mu) \xrightarrow{I} L_1(U^0, \mu) \xrightarrow{Q} F',
\]

where \( |P| \leq 1, |Q| \leq 1 \) and \( \mu(U^0) = 1 \).

**Proposition 1.** Suppose that \( T : E \to F \) is an integral mapping with a norm \( \sigma(T) = 1 \), and that for some unit element \( y_0 \in F' \) the element \( a_0 = T'y_0 \) is an extreme point of \( S(E') \). Then \( T \) is one-dimensional, and the image of \( T' \) is the line \( \{\lambda a_0 \} \) spanned by \( a_0 \).

**Proof.** We represent \( T \) as the composition (2). Then

\[
\langle x, T'y' \rangle = \langle QIPx, y' \rangle = \langle IPx, Q'y' \rangle = \int_{U^0} \langle x, a \rangle Q'y'd\mu
\]

for every \( y' \in F' \) and \( x \in E \). Hence, \( \langle x, a_0 \rangle = \int_{U^0} \langle x, a \rangle Q'y'd\mu \) for every \( x \in E \). For an arbitrary measurable set \( K \subseteq U^0 \) we have

\[
\langle x, a_0 \rangle = \int_K \langle x, a \rangle Q'y'd\mu + \int_{U^0 \setminus K} \langle x, a \rangle Q'y'd\mu
\]

We can consider the expressions

\[
a_1 = \int_K \langle x, a \rangle Q'y'd\mu; \quad a_2 = \int_{U^0 \setminus K} \langle x, a \rangle Q'y'd\mu
\]

as elements of the space \( E' \), and

\[
|a_1| \leq \mu(K); \quad |a_2| \leq \mu(U^0 \setminus K).
\]

Since \( a_0 = a_1 + a_2 \) and \( |a_1| + |a_2| \leq 1 \), from the condition

\[
a_0 \in \text{ext } S(E')
\]

we obtain \( a_1 = |a_1| a_0, a_2 = |a_2| a_0 \). Note that from here follows the relation

\[
|a_1| + |a_2| = 1
\]

and hence,

\[
|a_1| = \mu(K); \quad |a_2| = \mu(U^0 \setminus K).
\]