NECESSARY CONVERGENCE CONDITIONS FOR SERIES $\sum \xi_n S_n$ IN THE CASE OF IDENTICALLY DISTRIBUTED INDEPENDENT RANDOM QUANTITIES

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UDC 517

Necessary (in some cases also sufficient) conditions are obtained for convergence of the series $\sum \xi_n S_n$, where $\xi_n = \sum_{k=1}^{n} \xi_k$ are independent random quantities. The cases in which $\xi_k$ are symmetrical or identically distributed quantities are investigated in more detail.

In some probability theory problems the question arises as to whether the series

$$\sum S_n/n^\alpha,$$  \hspace{1cm} (1)

are convergent or divergent; $S_n = \sum_{k=1}^{n} \xi_k$, here denote independent random quantities, $\alpha > 0$.

The case of $\xi_k$ being independent and identically distributed was studied in a recently published article [1]; it seems also appropriate to assume here that $P(\xi \neq C) > 0 (\forall C)$, i.e., that the quantities $\xi_k$ are not equivalent to a constant. Let $R$ be a class of all such sequences. The following theorem was obtained in [1]: If $\alpha < 3/2$, then $\forall (\xi) \in R$ at least one of the series $\sum S_n/n^\alpha$, $\sum S_{-n}/n^\alpha$ diverges almost certainly (a.c.). In the above $S_n^+ = \max (S_n; 0)$, $S_n^- = \min (S_n; 0)$.

In connection with the above result the authors of [1] pose the following question: Is it correct to assert that for $\alpha < 3/2$ a stronger property holds, namely that the series (1) diverges a.c. for all $(\xi) \in R$ (and what is going to happen if $\alpha = 3/2$)?

In the present note convergence conditions are studied of the series

$$\sum a_n S_n$$  \hspace{1cm} (2)

with arbitrary coefficients $a_n$.

One says that $(\xi) \in S$ if $\xi_k$ are independent and symmetric. In this case it is possible to obtain a simple convergence test of the series (2) with $a_k \geq 0$ (Theorem 1).

Then the case of $(\xi) \in R$ is analyzed. In particular, the problem formulated in [1] is solved in the Corollaries 1 and 2.

Partial sums of the series (2) are expressed as linear combinations of $(\xi_k)$ by means of the formula

Translated from Matematicheskie Zametki, Vol. 20, No. 4, pp. 529-536, October, 1976.

Original article submitted March 25, 1975.
\[
\sum_{k=1}^{n} a_{k} S_{k} = \sum_{k=1}^{n} b_{kn} x_{k},
\]

\[
b_{kn} = \sum_{i=k}^{n} a_{i}.
\]

It is known that for symmetric independent random quantities the series \( \Sigma A_{k} x_{k} \) converges a.c. if and only if the series \( \Sigma A_{k} x_{k}^{2} \) converges a.c. (see, e.g., [2]). The latter follows from the fact that convergence of the series \( \Sigma A_{k} x_{k} \) implies that of the series \( \Sigma \pm A_{k} x_{k} \). Using a similar approach one can prove the following lemma.

**Lemma 1.** Let \((\xi_{k})\) be independent symmetric random variables and let \( \forall E \subseteq \Omega, P(E) > 0 \)

\[
\lim_{k \to \infty} \sum_{k=1}^{n} b_{kn} x_{k} < \infty \quad n, n., \lim_{n \to \infty} b_{kn} = 0.
\]

If

\[
\lim_{n \to \infty} \sum_{k=1}^{n} b_{kn} x_{k} \quad n, n., \\lim_{n \to \infty} b_{kn} = 0.
\]

then 3 \( k_{0} > 1 \) is such that

\[
\lim_{n \to \infty} \sum_{k=k_{0}}^{n} b_{kn} x_{k} < \infty.
\]

Proof. It follows from (6) and from the symmetry of the \((\xi_{k})\) that \( \forall E \subseteq (0, 1) \exists n_{1}, C_{4} \) and \( \forall E \subseteq \Omega, P(E) > 1 - \varepsilon \) such that

\[
|\sigma_{n}(\omega; t)|^{2} = \sum_{k=1}^{n} b_{kn} x_{k}(\omega) r_{k}(t) \leq C_{6}
\]

(\( \omega \subseteq E_{n}^{1}, n \geq n_{1} \)).

In the above \((r_{k}(t)) = (\text{sign} \sin 2^{k} \pi t)\) is the Rademacher system.

One now keeps 0 < \( \varepsilon_{0} < 1/8 \) constant and one introduces the notation

\[
E_{0} = \{\omega: \text{mes} [t \subseteq [0, 1] | |\sigma_{n}(\omega; t)|^{2} \leq C_{6} \} \geq 1/4\}
\]

\( G_{0} \) denotes the set in \( \Omega \times [0, 1] \), in which the estimate (7) is valid for \( \varepsilon = \varepsilon_{0} \); of course one has \( P(E_{0}) \geq 1/8 \). One now makes use of the following estimate: If \( A \subseteq [0, 1], \text{mes} A \geq 3/4 \) then*

\[
\int_{A} \left| \sum_{k} a_{k} r_{k}(t) \right|^{2} dt \geq 1/10 \sum_{k} a_{k}^{2} \quad (\forall n; a_{k}).
\]

By integrating (7) over \( G_{0} \) one obtains

\[
M \geq \sum_{k=1}^{n} b_{kn} \int_{E_{0}} x_{k}(\omega) P(d\omega), \quad n \geq n_{0}.
\]

One now has \( \int_{E_{0}} x_{k}(\omega) P(d\omega) \geq 1/2 C_{6}, \quad k \geq k_{0} \) by virtue of (5); the lemma has been proved.

**Lemma 2.** If for some values \( a_{k} \) the inequality

\[
\sum_{k=1}^{n} (\sum_{j=k}^{n} a_{j})^{2} < C \quad (\forall n \geq 1),
\]

is valid then

a) the series \( \Sigma a_{j} \) is convergent;

b) this estimate follows from the Khinchin inequality

\[
\left| \sum_{k} a_{k} x_{k} \right|^{2} \leq 3 \sum_{k} a_{k}^{2}.
\]

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