INTEGRATION OF MODULAR FORMS WITH THETA SERIES

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Dirichlet series, having holomorphic analytic continuation to the whole complex plane and satisfying a functional equation of standard type, are obtained by considering Rankin type integrals of products of elliptic modular forms for the group $\Gamma = \text{SL}_2(\mathbb{Z})$ by theta series of integral quadratic forms of determinant $\pm 1$. In a series of cases the constructed Dirichlet series are Mellin transforms of elliptic modular forms of higher weight than the initial forms.

1. Introduction

In [1], by the integration of theta series of indeterminate quadratic forms we have obtained Dirichlet series, having an analytic continuation to the whole complex plane and satisfying a functional equation with Gamma factors. Moreover, it turned out that in a series of cases the constructed Dirichlet series are Mellin transforms of elliptic modular forms. In this paper we show that Dirichlet series with similar properties and, in particular, Mellin transforms of elliptic modular forms are obtained also by considering integrals of a more general form, namely integrals of products of modular forms by theta series. In this way we obtain, in particular, integral transforms of elliptic modular forms into modular forms of higher weight; this may present interest in the theory of lifting of modular forms, especially if one succeeds to find the connection between these integral transforms and the Hecke operators.

For illustration we formulate some of the results.

**THEOREM 5.** Let

$$F(z) = \sum_{n=0}^{\infty} f(n) e^{2\pi i n z} \quad (z = x + iy; \ y > 0)$$

be a (holomorphic) modular form of integral weight $c \geq 0$ relative to the full modular group $\Gamma = \text{SL}_2(\mathbb{Z})$, and let $Q$ be an even (i.e., integral symmetric with an even main diagonal) matrix of order $m$. We assume that the determinant of $Q$ is equal to $\pm 1$ and that the quadratic form with the matrix $Q$, i.e., the form

$$q(x) = \frac{1}{2} x^T Q x,$$

where $x = (x_1, \ldots, x_m)$, has signature $(m - 1, 1)$. Assume finally that a real $m$-column $P$ satisfies the condition

$$P = -\frac{1}{2} x^T Q x > 0 \quad (1.2)$$

and that a complex $m$-column $\Omega$ satisfies the relations

$$x^T P \Omega = 0, \quad q(\Omega) = 0. \quad (1.3)$$

Then for each positive even number $a$ the Dirichlet series

$$A(s) = \zeta(s+2-a-c-m/2) \sum_{N \in \mathbb{Z}^m; N \neq 0} \frac{1}{N^a q(N)} \left| \sum_{\nu \in \mathbb{Z}^m} \right|^{1/2}$$

and, if $F$ is a cusp form (i.e., $f(0) = 0$), also the Dirichlet series

$$A(s) = \zeta(s+2-c-m/2) \sum_{N \in \mathbb{Z}^m; N \neq 0} \frac{1}{N^a q(N)} \left| \sum_{\nu \in \mathbb{Z}^m} \right|^{-s}$$

where $\zeta(s)$ is the Riemann zeta function, while $N$ runs through all integral nonzero $m$-columns on which the form $q$ assumes
nonnegative values, converge absolutely and uniformly in some right half-plane of the variable s; the function

$$\psi(s) = (2\pi)^{-s} \Gamma(s) \mathcal{D}(s),$$

where \(\Gamma(s)\) is the Gamma function, has an analytic continuation to the whole s-plane as an entire function, bounded in each vertical strip of finite width, and satisfies the functional equation

$$\psi(s + 2a + 2c - s) = \psi(s).$$

**THEOREM 6.** In the notations and assumptions of the preceding theorem, we assume, in addition, that the m-column \(P\) is integral and that

$$\tilde{\psi} = \frac{1}{\pi} \frac{\partial P^{-1}}{\partial \psi} = 1.$$

Then the Dirichlet series \(\mathcal{D}(s)\) can be written in the form

$$\mathcal{D}(s) = \sum_{n=1}^{\infty} \frac{1}{n(n) n^{-s}}$$

and its corresponding Fourier series (the inverse Mellin transform)

$$H(z) = \sum_{n=1}^{\infty} \frac{h(n)}{n} e^{i\pi n z}$$

is a modular cusp form of weight \(m + 2a + 2c - 2\) relative to the full modular group \(\Gamma\).

Theorems 5 and 6 will be proved in Sec. 4. In Sec. 5 from Theorem 6 we shall derive the following formula for the Ramanujan function \(\tau(n)\), defined by the expansion

$$\Delta(z) = e^{2\pi i x} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z})^{2n} = \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n z}. \tag{1.4}$$

**THEOREM 7.** Let \(Q_8\) be the matrix of an integral, positive definite quadratic form \(q \in \mathbb{Q}\) of 8 variables, of determinant 1, for example, of the form

$$q_8(x_1, \ldots, x_8) = \frac{1}{a} \sum_{r=1}^{8} x_r^2 + \frac{1}{b} \left( \sum_{r=1}^{8} x_r \right)^2 - x_1 x_8 - x_2 x_8,$$

and assume that the numbers \(\omega_1, \ldots, \omega_8 \in \mathbb{C}\) satisfy the relation

$$q_8(\omega_1, \ldots, \omega_8) = -1.$$

Then the Ramanujan function \(\tau(n)\) can be given by the formula

$$\tau(n) = \frac{1}{\sqrt{q}} \sum_{t \in \mathbb{Z}} \sum_{N \geq \Delta} \left( q_8(\Omega + t) \right)^\circ,$$ \tag{1.5}

where \(\Omega\) is a column with elements \(\omega_1, \ldots, \omega_8\).

In fact, this paper is a continuation of [1] and we shall use extensively the definitions and constructions contained there.

**Notations.** By \(\mathbb{N}, \mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}\) we denote the set of natural numbers, the set of nonnegative integers, the ring of integers, the field of real numbers, and the field of complex numbers, respectively. \(A_{m,n}\) is the set of all \(m \times n\) matrices with elements from \(A, A_{m} = A_{m}, A_{n} = A_{n}\). We denote by \(S_m\) the set of all symmetric matrices from \(A_{m,n}\), and by \(S^+_{m}\) and \(S^+_{m}\) the subsets of invertible and positive definite matrices from \(S_m\), respectively. If \(M\) is a matrix, then \(\text{t}M\) is the transposed matrix and we write

$$S[M] = \text{t}M \cdot S \cdot M,$$

provided the product of matrices in the right-hand side makes sense.