The following theorem is proved: if for all \( x \in \mathbb{Z}^n \) (\( x \neq 0 \)) one has \( |F(x)| \geq \mu > 0 \), where \( F(x) \) is a decomposable form of degree \( n \) of \( n \) variables, then, for \( n \geq 3 \), \( F(x) \) is proportional to an integral form.

1. Formulation of Results, Notations, and Definitions

1°. THEOREM A. Assume that there is given a form \( F(x) = f_1(x) \cdots f_n(x) \), where \( f_1, \ldots, f_n \) are linear real forms of \( n \) variables (\( n \geq 3 \)). If for all \( x \in \mathbb{Z}^n \) (\( x \neq 0 \)) we have \( |F(x)| \geq \mu > 0 \), then \( F(x) \) is proportional to an integral form.

From this theorem (see [1], [2]) there follows THEOREM B (Littlewood’s problem).

\[
\lim_{m \to \infty} m \left| \sin m \pi \alpha \cdot \sin m \pi \beta \right| = 0, \quad (m=1,2,\ldots) \text{ for } \alpha, \beta \in \mathbb{R}.
\]

2°. Notations and Definitions.

The elements of \( \mathbb{R}^n \) will be called points or vectors. By \( N(X) \) we denote the quantity \( |x_1 \cdots x_n| \), where \( x_1, \ldots, x_n \) are the coordinates of the point \( X \in \mathbb{R}^n \). The quantity \( N(\Lambda) = \inf \{ N(Y) : Y \in \Lambda, (Y \neq 0) \} \) will be called the normed minimum of the lattice \( \Lambda \subset \mathbb{R}^n \). Two lattices \( \Lambda, \Lambda' \subset \mathbb{R}^n \) will be said to be similar, written \( \Lambda \sim \Lambda' \), if \( \Lambda' = \mathcal{A} \Lambda \) for some diagonal matrix \( \mathcal{A} \).

A lattice \( \Lambda \subset \mathbb{R}^n \) will be said to be algebraic or an \( a \)-lattice if it is similar to the lattice of the complete module of a purely real algebraic field of degree \( n \); otherwise, the lattice \( \Lambda \) will be called a \( b \)-lattice.

By \( D \) we denote the full collection of diagonal unimodular matrices with positive elements. By \( G, \Gamma \) we denote the groups \( SL(n, \mathbb{R}), SL(n, \mathbb{Z}) \), respectively.

By \( G(\Lambda) \) we denote the stabilizer of \( \Lambda \).

In all the subsequent presentation (unless otherwise specified), we consider unimodular lattices such that \( \Lambda : \mathbb{Z}^n \) and \( d(\Lambda) = |\det A| \).

The space of unimodular lattices will be identified in the natural manner with \( G/\Gamma \).

3°. Mahler’s compactness criterion allows us to formulate Theorem A and Theorem B in terms of lattices.

THEOREM A’. If the orbit \( D \Lambda (\Lambda \subset \mathbb{R}^n, \mu > 0) \) is relatively compact in \( G/\Gamma \), then \( \Lambda \) is an \( a \)-lattice.

THEOREM B’. If \( \alpha, \beta \in \mathbb{R} \), then for a preassigned \( \varepsilon > 0 \) there exist integers \( p_1, p_2, q \neq 0 \) such that \( |q(p_1 + q_1 \alpha - p_2 q_2 \beta)| < \varepsilon \).

In this note we present the proof of Theorem A’. At the conclusion we present the proof of Theorem B”, from which (as a special case) we obtain the solution of the Littlewood problem.

2. Continuous Sets

1°. Orbits of Lattices.

LEMMA 1. Assume that there is given a subgroup \( D \subseteq \mathbb{Z} \); if the orbit of \( \Lambda \) is relatively compact in \( G/T \), then for any \( \lambda \in \mathbb{Z}^{n} \) we have:

(i) \( N(\lambda) > N(\Lambda) = \mu > 0 \),

(ii) \( \mathbb{Z}^{n} \in \mathbb{Z}^{n} \),

(iii) if \( n \geq 3 \) and \( \Lambda \) is a b-lattice, then \( \Lambda' \) is a b-lattice.

Proof. (i) and (ii) are obvious. (iii) In the case \( \Lambda = \Lambda' \), statement (iii) is obvious. We assume that \( \Lambda' \) is a b-lattice. Since the orbit DA is dense in \( \mathbb{Z}^{n} \), for a given \( \epsilon > 0 \) there exists \( \lambda \in \mathbb{Z}^{n} \) such that \( ||A_{\lambda} - \lambda|| < \epsilon \). Since \( N(\lambda') = N(\lambda) \), by the isolation theorem (see [3]) we obtain \( N(\lambda') < N(\lambda) \), contradicting statement (i). Lemma 1 is proved.

Remark. For \( n = 2 \), statement (iii) is not always true. For example, set \( \Lambda = \mathbb{Z}^{2} \), where \( \Lambda = \{ x + y \mid x, y, x, y \in \mathbb{Z} \} \). Then (i), (ii) are satisfied but (iii) does not hold since \( \mathbb{Z}^{n} \) is not a b-lattice. For each indicated \( \lambda \). We shall assume that \( \Lambda \) is a b-lattice.

2. Cycles of Lattices.

Let DA be as in Lemma 1. If for \( \Lambda \in \mathbb{Z}^{n} \) we have the equality \( \mathbb{Z}^{n} = \mathbb{Z}^{n} \) for all \( \Lambda \in \mathbb{Z}^{n} \), then the set \( \mathbb{Z}^{n} \) will be called a cycle and we write \( \{ \mathbb{Z}^{n} \} \).

LEMMA 2. The set \( \mathbb{Z}^{n} \) contains \( \Lambda_{0} \) such that \( \mathbb{Z}^{n} \subseteq \mathbb{Z}^{n} \).

Proof. From Lemma 1(ii) for \( \Lambda \in \mathbb{Z}^{n} \) there follows the inclusion \( \mathbb{Z}^{n} \subseteq \mathbb{Z}^{n} \). If for all \( \Lambda \) from \( \mathbb{Z}^{n} \) we have \( \mathbb{Z}^{n} = \mathbb{Z}^{n} \), then \( \mathbb{Z}^{n} = \mathbb{Z}^{n} \) and the lemma is proved. Assume that for some \( \Lambda \in \mathbb{Z}^{n} \) we have the strict inclusion \( \mathbb{Z}^{n} \neq \mathbb{Z}^{n} \). Carrying out a similar argument for the set \( \mathbb{Z}^{n} \), we obtain the strict inclusion \( \mathbb{Z}^{n} \neq \mathbb{Z}^{n} \). At the i-th step we shall obtain the strict inclusion \( \mathbb{Z}^{n} \neq \mathbb{Z}^{n} \). If the number of such steps is finite, then the lemma is proved. Otherwise, we shall have an infinite sequence of strict inclusions: \( \mathbb{Z}^{n} \neq \mathbb{Z}^{n} \), \( \mathbb{Z}^{n} \neq \mathbb{Z}^{n} \), ..., \( \mathbb{Z}^{n} \neq \mathbb{Z}^{n} \). Let \( \Lambda_{0} \) be one of the limit points of the infinite sequence \( \Lambda_{0}, \Lambda_{1}, ..., \Lambda_{n-1} \), \( \Lambda_{n} \). We have \( \mathbb{Z}^{n} = \mathbb{Z}^{n} \). Lemma 2 is proved.

For the sake of convenience (if this does not lead to confusion) the cycles \( \{ \mathbb{Z}^{n} \} \) will be denoted by \( \{ D \} \).

3. Discrete Sets

The Fundamental Properties of the Lattice \( \Lambda \subseteq \mathbb{R}^{n} \) for \( N(\Lambda) = \mu > 0 \).

LEMMA 3. If the lattice \( \Lambda \subseteq \mathbb{R}^{n} \) we have \( N(\Lambda) = \mu > 0 \), then for reciprocal lattice \( \Lambda^{*} \) we have \( N(\Lambda^{*}) = \mu^{*} > 0 \).

Proof. We assume that \( N(\Lambda^{*}) = 0 \). Then for a preassigned \( \epsilon > 0 \), one can find \( Y_{\epsilon} \in \Lambda^{*} \) such that \( N(Y_{\epsilon}) < \epsilon \). From here we obtain the existence of \( Y_{\epsilon} \in \Lambda_{n-1} \subseteq \mathbb{R}^{n-1} \) of length \( \epsilon = \rho \), \( \rho = 1 \), which is not possible. (Here \( \rho_{n-1} \) is the volume of the \( (n-1) \)-dimensional ball with radius equal to 1.) Lemma 3 is proved.

COROLLARY 1. If lattice \( \Lambda \) is from Lemma 3, \( \Lambda \) is a matrix basis of \( \Lambda \) (so that \( \Lambda = \Lambda^{*}^{T} \)), then \( \Lambda^{*} \) is a matrix basis of \( \Lambda^{*} \) and, therefore, all the elements of the matrix \( \Lambda^{-1} \) are different from zero.

COROLLARY 2. Let \( Y_{1}, ..., Y_{n-1} \) be \( n-1 \) linearly independent vectors from \( \Lambda \). We consider the plane \( \Delta_{n-2} : \{ \alpha_{1} Y_{1} + ... + \alpha_{n-1} Y_{n-1} \mid \alpha_{1} + ... + \alpha_{n-1} = 1 \} \) and the point \( (p_{1}, ..., p_{n-1}, q_{n}) \in \Delta_{n-2} \). The point \( (p_{1}, ..., p_{n-1}, q_{n}) \) lies in \( \Delta_{n-2} \) if and only if \( p = q = 1 \). It follows from Corollary 1.

By \( p(a_{1}, ..., a_{n}) \) we denote the set \( \{ \alpha_{1} Y_{1} + ... + \alpha_{n-1} Y_{n-1} \mid 0 < \alpha_{1}, ..., \alpha_{n-1} < 1 \} \) and its volume.

LEMMA 4. There exist constants \( v_{0}, v_{1} \) such that: (i) \( \Lambda \cap p = \varnothing \) for \( v_{0} > \lim_{i} a_{i} \); (ii) \( \Lambda \cap p = \varnothing \) for \( v_{1} > \lim_{i} a_{i} \), and \( v_{0}, v_{1} \) depend only on \( \mu = N(\Lambda) \).

Proof. Since for any \( \{ D \} \) in the lattice \( \mathbb{Z}^{n} \), the length of any vector (except 0) is at least \( \mu^{\text{th}} \sqrt{n} \), it follows that (i) is satisfied for \( v_{0} = \mu \). For the proof of (ii) we consider the cylinder:

\[ \{ x \in \mathbb{R}^{n} \mid \langle x, x \rangle - \frac{1}{\mu^{\text{th}} \sqrt{n}} (\sum_{i=1}^{n} x_{i})^{2} < 0 \} \]

of volume \( V(\mu, h) = \mu^{\text{th}} \sqrt{n} - \frac{1}{\mu^{\text{th}} \sqrt{n}} \cdot (1 + \frac{n-1}{2}) \cdot h \), for \( 0 < h \leq \mu^{\text{th}} \sqrt{n} - \frac{1}{\mu^{\text{th}} \sqrt{n}} \) and also \( h \), such that \( V(\mu, h) = m \cdot \mu^{\text{th}} \). Then (Minkowski's theorem) \( C(r, h) \) will contain at least \( m \) points of \( \Lambda \). All these points are from the first orthant.