DIFFRACTION IN A NONLINEAR DEFOCUSING MEDIUM

R. F. Bikbaev

Diffraction by a semitransparent screen is considered in the framework of the nonlinear Schrödinger equation. The case of a defocusing medium is investigated. The method used represents a synthesis of the method of Riemann's matrix problem and the technique of Whitham's deformations of spectral curves.

Introduction. Application of the inverse scattering problem method (ISPM) made it possible [1] to achieve substantial progress in the study of nonlinear partial differential equations. Equations integrable by the ISPM emerge (as model equations) in various physical (and mathematical!) applications. As a rule, in order to study such models analytically, simple "fast decreasing" or periodic boundary conditions are considered. However, there are several physically interesting problems that can be simulated by integral equations with nontrivial boundary conditions.

In the present article we consider one of the fundamental systems of the ISPM, namely the nonlinear Schrödinger equation (NS)

\[ i_\hbar \phi_t + \phi_{xx} - 2\|q\|^2\phi = 0, \quad \hbar = 1, \quad x, z \in \mathbb{R}, \quad q \in \mathbb{C} \]  \hfill (1)

with a boundary condition \( q(x, 0) \) that converges fast (for simplicity, in the sense of the Schwarz class) to two distinct constants as \( x \to \pm \infty \),

\[ q(x, 0) \to q_\pm \quad \text{as} \quad x \to \pm \infty, \quad 0 < |q_-| < |q_+|. \]  \hfill (2)

In particular, we are interested in boundary conditions of the "step" type:

\[ q(x, 0) = q_- + \delta(x)(q_+ - q_-), \quad \delta(x) = \frac{1 + \sign x}{2}. \]  \hfill (3)

Equation (1) with the boundary condition (3) describes the following problem in nonlinear optics (see [2], p. 356).

Along the \( z \)-axis there propagates a flat stationary electromagnetic wave and \( q(x, z) \) describes the modulation of the complex envelope of the wave. The half-plane \( z \geq 0 \) is filled with a nonlinear defocusing (\( \hbar = 1 \)) medium. There is a semitransparent screen in the region \( x < 0 \) of the plane \( z = 0 \), so that the radiation intensity \( I = |q|^2 \) immediately behind the screen is less than in the "exposed" region \( z = 0, x > 0 \).

We investigate the (Fraunhofer) diffraction of a wave by the screen in the "far" region \( z >> 1, x \in \mathbb{R} \). From the mathematical point of view this is a problem concerning the asymptotic behavior of the solution \( q(x, z) \) of the system (1), (3) as \( z \to +\infty \).

We remark that S. V. Manakov was the first to consider a similar problem for the NS model in the simpler case where \( q(x, 0) \) is a function with bounded support (see [1], p. 247). The case where \( |q_+| = |q_-| = \text{const} \), which comes next as far as complexity is concerned, was investigated in [3] (see also [4]).

In the case of boundary conditions (3) (or (2)) the behavior of \( q(x, z) \) is of much more complex nature than in [1] or [3], and there are five pairwise disjoint regions \( \Omega_i, i = 1, \ldots, 5 \) in the \( x, z \)-plane such that the form of the solution is completely different in each of the regions as \( z \to +\infty \). To be more specific, we introduce the variable \( \xi = x/z \) and we consider the following regions \( \Omega_i \), where \( i = 1, \ldots, 5 \):

\[ \xi \in \Omega_i \iff \begin{cases} \xi < \xi_1 - \epsilon & \text{if} \quad i = 1, \\ \xi_{i-1} + \epsilon < \xi < \xi_i - \epsilon & \text{if} \quad 1 < i < 5, \\ \xi > \xi_4 + \epsilon & \text{if} \quad i = 5. \end{cases} \]  \hfill (4)

Here \( \epsilon > 0 \) is a small fixed constant. The constants \( \xi_i \) depend on \( \alpha_+ = |q_+| \) and \( \alpha_- = |q_-| \) only, and have the form

\[ \begin{align*} 
\xi_1 &= (-4\alpha_+^2 + 2\alpha_-^2)/\alpha_+, \\
\xi_2 &= (-3\alpha_+^2 + \alpha_-^2 - 2\alpha_+\alpha_-)/(\alpha_+ + \alpha_-), \\
\xi_3 &= (3\alpha_+^2 - \alpha_-^2 + 2\alpha_+\alpha_-)/(\alpha_+ + \alpha_-), \\
\xi_4 &= 2\alpha_. 
\end{align*} \]
Let us note that $\xi_1, \xi_2 < 0$ and $\xi_4 > 0$. $\xi$ can be interpreted in a natural way as the "diffraction angle." It turns out that the intensity of diffracted radiation $I = |q(x,z)|^2$ as a function of the diffraction angle has the form shown in Fig. 1 in the limit as $z \to +\infty$. (It is assumed that there are no solitons.) The intensity $I = |q|^2$ is constant in $\Omega_1$ and $\Omega_5$. If there are no solitons, then $I = \left(\frac{\alpha_+ + \alpha_-}{2}\right)^2 = \text{const}$ in $\Omega_3$.

**REMARK 1.** For solitons to be absent in the case of a boundary condition of the step type (3) it is necessary and sufficient that the inequality

$$\arg \left( |q_-| + i \sqrt{|q_+|^2 - |q_-|^2} \right) > \arg \left( q_- / q_+ \right)$$

be satisfied. In particular, there are no solitons if $q_+, q_- \in \mathbb{R}_+$. If the inequality sign in (5) is changed, there must be exactly one soliton in $\Omega_3$ (see below for details).

$I = I(\xi)$ is a quadratic function in $\Omega_4$. Finally, $I(\xi)$ oscillates in $\Omega_2$. In the intermediate $\varepsilon$-zones between the regions $\Omega_i$ (see (4)) the asymptotic behavior has a more complex character. However, in the leading order with respect to $\varepsilon^{-1}$ the asymptotics can be obtained by a limiting passage from the "external" domains $\Omega_i$.

To prove the results stated and to make them more rigorous, we apply the ISPM technique in the form proposed in [5] and [6]. Let us note [5] that in a similar way one can investigate another physically interesting example, namely the disintegration of an initial discontinuity in the theory of the Kortweg-de Vries equation ([1], p. 260).

1. The spectral problem, scattering data, and Whitham's deformation of the spectrum. The NS equation (1) can be written as the compatibility condition for the system $\partial_\varepsilon \psi = L \varepsilon, \partial_\varepsilon \psi = A \psi$ (see [1]). We define the Jost solutions $\psi_+(x, z, \lambda)$ and $\psi_-(x, z, \lambda)$ as the joint solutions of the system determined by the asymptotics

$$\psi_\pm \to \exp(-\text{i}k_{\pm}(x + 2\lambda z)) \exp(-\text{i}\sigma_3 |q_{\pm}|^2) \left( \frac{1}{\lambda - k_{\pm}} \right)$$

as $x \to \pm \infty$, where $\sigma_3 = \text{diag}(1, -1)$. The above asymptotic functions are consistent with the boundary conditions (2) and with the form of the Lax operators $L - A$. In these formulas $\lambda$ is an auxiliary spectral parameter and $k_{\pm} = k_{\pm}(\lambda) = (\lambda^2 - \alpha_{\pm}^2)^{1/2}$ is a single-valued algebraic function of $\lambda$ defined on the Riemann surface $\Gamma_{\pm}$ obtained by pasting together two copies of the complex plane $\mathbb{C}$ with cuts along the spectrum $E_{\pm} = (-\infty, -\alpha_{\pm}] \cup [\alpha_{\pm}, +\infty)$. We shall assume that $\text{Im} k_{\pm}(\lambda) > 0$ on the upper sheet $\Gamma_{\pm}^U$ of $\Gamma_{\pm}$. If $\lambda \in E_-$, then the Jost functions are connected by the scattering relation

$$\psi_+(P) = \psi_-(P)a(P) + \psi_-(\sigma P)b(P).$$

In this formula $P = (\lambda, k_{\pm})$ is a point on $\Gamma_{\pm}$ and $\sigma$ is an involution defined as the transposition of the sheets of $\Gamma_{\pm}$. The scalar functions $a(P)$ and $b(P)$ are independent of $x, z$. $a(P)$ admits analytic continuation from the cut $E_-$ to the lower sheet $\Gamma_{\pm}^L$ of $\Gamma_{\pm}$. A more detailed analysis of the properties of the scattering data $a(P)$ and $b(P)$ can be carried out analogously as in [5] and [6] and is omitted for the sake of conciseness.

In what follows we restrict our considerations to the case without solitons, i.e., we shall assume that

$a(P)$ does not vanish for $P \in \Gamma_{\pm}^L$.

Let us note that for the boundary conditions (3) this requirement is equivalent to (5), and so it can be easily controlled. Indeed, in this case the scattering data can be explicitly computed:

$$a(P) = [(\lambda + k_-)q_+ - (\lambda - k_+)q_-] / 2k_- q_+, \quad b(P) = 1 - a(P).$$