We investigate the approximation of functions by Bernstein polynomials. We prove that

\[ r_{[0,1]}(f, B_n(f)) \leq \nu_f \left( s \sqrt{\ln n} \right) + o \left( \sqrt{\ln n} \right), \]

where \( r_{[0,1]}(f, B_n(f)) \) is the Hausdorff distance between the functions \( f(x) \) and \( B_n(f; x) \) in \([0,1]\),

\[ \nu_f(s) = \frac{1}{2} \sup_{x_1, x_2 \in [0,1]} \{ \sup_{x_1, x_2 \in [0,1]} \| f(x_1) - f(x_2) \| + \| f(x_2) - f(x_1) \| - \| f(x_1) - f(x_2) \| \} \]

is the modulus of nonmonotonicity of \( f(x) \). The bound (1) is of better order than that obtained by Sendov. We show that the order of (1) cannot be improved.

The rate of convergence of Bernstein polynomials in the Hausdorff metric was investigated by Sendov [1]. We quote the basic result.

First, we recall some definitions [2]. Let \( f(x) \) be a function defined and bounded in an interval \( \Delta \). The complemented graph \( \bar{f} \) of \( f(x) \) is the minimal point set in the \( xOy \) plane which is closed and convex with respect to the \( y \)-axis and which contains the graph of \( f(x) \). The Hausdorff distance with parameter \( s > 0 \) between the functions \( f(x) \) and \( g(x) \), specified and bounded in \( \Delta \) is defined as follows:

\[ r_{\Delta}(f, g; s) = \max \{ \max_{x \in \bar{f}} \min_{y \in \bar{g}} \rho(X, Y), \max_{y \in \bar{g}} \min_{x \in \bar{f}} \rho(X, Y) \}, \]

where

\[ \rho(X, Y) = \rho \left( X \left( x_1, y_1 \right), Y \left( x_2, y_2 \right) \right) = \max \left\{ \frac{1}{2} \left| x_1 - x_2 \right|, \left| y_1 - y_2 \right| \right\}. \]

In what follows we shall assume that \( s = 1 \) and then we shall write \( r_{\Delta}(f, g; s) \) as \( r_{\Delta}(f, g) \). All the results obtained can easily be extended to the case of arbitrary fixed \( s > 0 \). The function

\[ \mu_f(\delta) = \frac{1}{2} \sup_{x_1, x_2 \in [0,1]} \{ \sup_{\epsilon_1, \epsilon_2 \in [0,1]} \left( |f(x_1) - f(x_2)| + |f(x_2) - f(x_1)| - |f(x_1) - f(x_2)| \right) \}

is called the modulus of nonmonotonicity of \( f(x) \) in \( \Delta \). Let \( R_{\Delta} \) denote the class of functions \( f(x) \), defined in \( \Delta \), such that \( \lim_{\delta \to 0} \mu_f(\delta) = 0 \). The class \( R_{\Delta} \) coincides with the class of functions \( f(x) \) which do not have
second-order points of discontinuity and for which \( f(x) \) lies between \( f(x + 0) \) and \( f(x - 0) \) for each \( x \in \Delta \) (with the exception of the ends of the interval \( \Delta \)).

It was proved in [1] (see also [2]) that for any function \( f(x) \in \mathcal{R}_{[0,1]} \), \( \sup_{x \in [0,1]} |f(x)| \leq M \), we have

\[
\rho_{[0,1]}(f, B_n(f)) \leq C_1 \left( n^{-1} + \frac{1}{\ln n} \right) + C_2 n^{-1} \frac{1}{\ln n},
\]

where \( C_1 > 0 \) is a constant depending only on \( M \). In [2] the question was posed: is the bound (1) exact in order? In this paper we give a negative answer to that question and find a bound which is exact in order for the approximation of a function \( f(x) \in \mathcal{R}_{[0,1]} \) by polynomials \( B_n(f, x) \) in the Hausdorff metric.

**Theorem 1.** If \( f(x) \in \mathcal{R}_{[0,1]} \), \( \sup_{x \in [0,1]} |f(x)| \leq M \), for any natural number \( n \geq 2 \), we have

\[
\rho_{[0,1]}(f, B_n(f)) \leq C_1 \left( c_2 \sqrt{\frac{1}{x} n} \right) + c_3 \sqrt{\frac{\ln n}{n}},
\]

where \( c_2 \) is an absolute constant \((0 < c_2 \leq 4)\), \( c_3 > 0 \) is a constant depending only on \( M \).

**Proof.** For any natural numbers \( m \) and \( n \) we put

\[
W_{m,n}(x) = \sum_{i=0}^{n} \binom{n}{i} x^i (1-x)^{n-i}.
\]

It was established in [1] and [2] that for any \( \delta > 0 \) and any natural number \( k \), we have

\[
\rho_{[0,1]}(f, B_n(f)) \leq \max \left\{ 2\delta, 2\delta (4k) + 2M5^{-2k} \sup_{x \in [0,1]} W_{2k,n}(x) \right\}.
\]

Bernstein [3] proved that

\[
|W_{m,n}(x)| \leq \frac{m!}{n^{m/2}} \exp \left( |x(1-x)| \right)
\]

for \( |x| \leq \sqrt{n} \), \( |1-x| \leq \sqrt{n} \).

(4)

However, if \( x \in [0,1] \), we can obtain a bound which is more exact in some sense than (4). In particular, we shall prove that for all natural numbers \( m \geq 1 \) and \( n \geq 2 \), and for all \( x \in [0,1] \), we have

\[
|W_{m,n}(x)| \leq \frac{m^n!}{(n \ln n)^{m/2}}.
\]

(5)

We note first that the following identity [3] holds:

\[
\sum_{m=0}^{\infty} \frac{W_{m,n}(x)m^n}{m!} = \exp (-ux) \left( 1 + x \left( \exp \left( \frac{u}{n} \right) - 1 \right) \right)^n.
\]

(6)

Let \( C \) denote the circle \( |z| = (n \ln n)^{1/2} \). Using Cauchy's integral formula from (6), we obtain

\[
\frac{1}{m!} W_{m,n}(x) = \frac{1}{2\pi i} \int_{C} \frac{\varphi_n(z,x)^n}{z^{m+1}} dz,
\]

(7)

where \( \varphi_n(z,x) = \exp \left( -\frac{ux}{n} \right) \left\{ 1 + x \left( \exp \left( \frac{z}{n} \right) - 1 \right) \right\} \). Let \( A \) denote the maximum of the modulus of \( \varphi_n(z,x) \) on \( C \). Since

\[
\varphi_n(z,x) = 1 + x (1-x) \frac{z}{n} k \left( -\frac{z}{n} \right)^{k-1} \cdot \left( (1-x)^{k-1} - x^{k-1} \right),
\]

then

\[
A \leq 1 + \frac{1}{4n} \sum_{k=3}^{\infty} \frac{1}{k!} < 1 + \frac{1}{4n} \frac{1}{3n} \quad \text{for all} \quad x \in [0,1].
\]

(8)