FUNCTIONS OF BOUNDED p-VARIATION WITH GIVEN
ORDER OF MODULUS OF p-CONTINUITY

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We construct continuous functions for which the modulus of p-continuity tends to zero with
given order in Wiener's sense

$$V^p(\delta; f) = \sup \sum_i |f(x_i) - f(x_{i-1})|^p \quad (p > 1)$$

(the upper bound is taken over partitions satisfying the condition $x_i - x_{i-1} \leq \delta$).

Wiener defines [1] the p-variation of a function in $[a, b]$ as follows:

$$V^p(\delta; f) = \sup \sum_i |f(x_i) - f(x_{i-1})|^p,$$

where the upper bound is taken over partitions of $[a, b]$ satisfying the condition $x_i - x_{i-1} \leq \delta$. If $p > 1$, $V^p(\delta; f) \to 0$ as $\delta \to 0$ for functions of a nontrivial class; we may call such functions p-continuous and say that

$$\omega_{-1/p}(\delta; f) = \{V^p(\delta; f)\}^{1/p}$$

is the modulus of p-continuity. The modulus is denoted in this way because it is a comparison function of
order $1-1/p$, which follows from the easily verifiable inequalities

$$\omega_{-1/p}(\delta; f) \leq n^{1-1/p} \omega_{-1/(np)}(\delta; f) \quad (n = 1, 2, \ldots). \quad (1)$$

The property of p-continuity is intermediate between the properties of continuity ($p = \infty$) and absolute con-
tinuity ($p = 1$; the modulus of absolute continuity can be taken as the upper bound of the variation of the func-
tion on sets $e$ of measure $m_e \leq \delta$).

We also assume that $1 < p < \infty$ and that $[a, b] = [0, 1]$ (we can transform to any interval by a linear
transformation of the variable).

THEOREM 1. Let $\omega(\delta)$ be a nondecreasing function in $[0, 1]$ ($\equiv 0$). For there to be a continuous func-
tion such that

$$\omega_{-1/p}(\delta; f) \times \omega(\delta),$$

it is necessary and sufficient that the product $\delta^{1/p-1} \omega(\delta)$ should almost not increase, i.e., for some con-
cstant $C > 0$,

$$\delta^{1/p-1} \omega(\delta) \leq C \delta^{1/p-1} \omega(\delta),$$

if $\delta \leq h$.

Proof. It follows from (1) that the modulus of p-continuity, and hence, $\omega(\delta)$, satisfies the necessary
condition.
To prove sufficiency we transform to the function
\[ \omega^*(\delta) = \delta^{1-1/p} \sup_{h>0} h^{1/p-1} \omega(h), \]
which, as is easily verified, is also nondecreasing; the corresponding product does not increase and
\[ \omega(\delta) \leq \omega^*(\delta) \leq C \omega(\delta). \]
Assuming \( \omega(1) = 1 \) (which can be achieved by normalizing \( \omega \)) and putting \( \omega_n = \omega^*(3^{-n}) \), we derive the existence of \( f \) from the following

**THEOREM 2.** Let \( 1 = \omega_0 \geq \omega_1 \geq \ldots \) and \( \omega_n \leq 3^{-1} \omega_{n+1} \). Then there is a function \( f \) which is continuous in \([0, 1]\) and such that
\[ V^p(3^{-n}; f) = \omega_n \quad (n = 0, 1, \ldots). \]

**Note.** By (1) and the monotonicity of the modulus of \( p \)-continuity, the conditions of the theorem are necessary. Continuity is relevant when \( \omega_n \) does not tend to zero.

To prove the theorem, we use the following simple lemma.

**LEMMA.** Let \( 3^{-p} \leq \theta \leq 1 \). It is asserted that when \( x \geq 0, y \geq 0 \) the equation
\[ 2(y + x/3)^p + |2y - x/3|^p = 6x^p \]
has the unique solution \( y = r(\theta)x \); the function \( r(\theta) \) increases, \( 0 \leq r(\theta) < 2/3 \), from which \( |2r(\theta) - (\theta/3)| \leq r(\theta) + (\theta/3) \).

**Proof.** For \( x = 0 \) the solution is \( y = 0 \), which agrees with the equation \( y = r(\theta)x \) for any \( r(\theta) \). Hence, it is sufficient to prove the equation for \( x > 0 \).

Dividing both sides of the equation by \( x^p \), we obtain the following equation for \( r(\theta) \):
\[ f(r, \theta) \equiv 2(r + \frac{1}{3})^p + |2r - \frac{1}{3}|^p - \theta = 0. \]
We have
\[ f(0, \theta) = 3^{-p+1} - \theta \leq 0 \text{ and } f(r, \theta) > 0, \]
if \( r \geq \frac{1}{3} \). Consequently, the solution \( r(\theta) \) exists, \( 0 \leq r(\theta) < \frac{1}{3} \), where \( r(\theta) = 0 \) only when \( \theta = 3^{-p} \). The solution is unique, since \( \partial f / \partial r > 0 \) for \( r > 0(2r \neq \frac{1}{3}) \):
\[ \frac{\partial f}{\partial r} = 2p(r + \frac{1}{3})^{p-1} + 2p|2r - \frac{1}{3}|^{p-1} \text{ sign } (2r - \frac{1}{3}), \]
from which \( \partial f / \partial r > 0 \), if \( 2r > \frac{1}{3} \); but if \( 2r < \frac{1}{3} \), then \( r + \frac{1}{3} > \frac{1}{3} - 2r \) for \( r > 0 \), which means that \( \partial f / \partial r > 0 \) in that case also.

Finally,
\[ r'(0) = - \frac{\partial f}{\partial \theta} \bigg| \frac{\partial f}{\partial r} > 0 \]
and, consequently, \( r(\theta) \) increases. The lemma is proved.

We turn to the construction of the function whose existence is postulated in the theorem. Fundamental to this construction are the continuous broken lines \( \varphi(a, b; x) \), equal to zero for \( x \leq a \) and \( x \geq b \), +1 and -1, respectively, at points \( a + \frac{1}{3}(b-a) \) and \( a + \frac{2}{3}(b-a) \), and linear elsewhere. We construct the unknown function in the interval \([0, 1]\) in the form of a series, defining its terms \( f_k \) by induction.

Let
\[ \theta_n = \omega_{n+1}/\omega_n \quad (n = 0, 1, \ldots), \]
Put
\[ f_n(x) = x, \quad s_n = f_n, \quad \Delta s_n = s_n(1) - s_n(0) = 1. \]