In this paper we study the role of functioning axioms on the deductive power of the system obtained from the Zermelo-Fraenkel ZF system by the introduction of \( \varepsilon \)-terms with the possibility of using them as a scheme for the substitution axiom. It is proved that if the system has a founding axiom the introduction of \( \varepsilon \)-terms does not extend the class of ZF theorems, while if the founding axiom is absent, there is an extension of the ZF theorems.

Let ZF denote a Zermelo-Fraenkel system with choice axiom and founding axiom. \( \text{ZF}(\varepsilon) \) is the system obtained from ZF by the introduction of the \( \varepsilon \)-terms \( \varepsilon x A(x) \) with \( \varepsilon \)-axioms (e1) and (e2) and the possibility of using expressions containing \( \varepsilon \)-terms in the substitution axioms:

\[
\begin{align*}
\text{(e1)} & \quad \exists x \ A(x) \rightarrow A(\varepsilon x A(x)), \\
\text{(e2)} & \quad \forall x (A(x) \leftrightarrow B(x)) \rightarrow \varepsilon x A(x) = \varepsilon x B(x).
\end{align*}
\]

In a recently published monograph [1] devoted to the \( \varepsilon \)-symbol the following problem was posed and said to be unsolved. Is there an expression \( A \), not containing \( \varepsilon \)-terms, provable in \( \text{ZF}(\varepsilon) \) and not provable in ZF? A negative answer, without proof, is given, however, in [2] (p. 149). As far as the author knows no proof has appeared in print.

In this note we prove

THEOREM 1. Let \( A \) be an arbitrary closed expression not containing \( \varepsilon \)-terms. Then

\[
\text{ZF}^{(\varepsilon)} \vdash A \quad \text{and} \quad \text{ZF} \not\vdash A.
\]

Let \( \text{ZF} \) denote ZF without founding axiom, but with the choice axiom. \( \text{ZF}^{(\varepsilon)} \) denotes \( \text{ZF}(\varepsilon) \) without founding axiom. For such theories Theorem 1 does not hold.

THEOREM 2. There is a closed expression \( A \), not containing \( \varepsilon \)-terms, provable in \( \text{ZF}^{(\varepsilon)} \) and not provable in \( \text{ZF} \).

Proof of Theorem 1. Consider the theory \( \text{ZF}(\sigma) \) obtained by the introduction into the language of ZF of a two-place predicate symbol \( \sigma \) with axioms

\[
\begin{align*}
\text{(s1)} & \quad \forall x \forall y \exists z (\sigma(xy) \land \sigma(xy) \rightarrow y = z), \\
\text{(s2)} & \quad \forall x (x \neq \phi \rightarrow \exists y (\rho x \land \sigma(xy)))
\end{align*}
\]

and the possibility of using \( \sigma \) in the substitution axiom. We know that \( \text{ZF}(\sigma) \) contains the same theorems formulated in the language of ZF as the theory \( \text{ZF}^{(\varepsilon)} \). This follows because in the theory \( \text{ZF}(\sigma) \) the \( \varepsilon \)-term \( \varepsilon x A(x) \) can be defined explicitly:

\[
y = \varepsilon x A(x) = \sigma((x, A(x) \land \forall t (\text{rang}(t) < \text{rang}(x) \rightarrow \neg A(t))), y).
\]
We use the method of constraint to transform proof in $ZF^{(σ)}$ into proof in $ZF$. Let the sign $≡$ denote "what is on the left is the notation for what is on the right," i.e.,

$$x ∈ P \equiv Func(x) \setminus \forall t (t ∈ φ \setminus tcD(x) → x(t) st),$$

$$p \vdash (xφy) ≡ (xφy),$$

$$p \vdash (x = y) \equiv (x = y),$$

$$p \vdash σ(xφy) ≡ (xy) φP,$$

$$p \vdash \forall A \equiv \exists q (q ∈ P \setminus q ⊃ p \setminus q \vdash A),$$

$$p \vdash (A \setminus B) \equiv p \vdash A \setminus p \vdash B,$$

$$p \vdash (A → B) \equiv ∀q (q ∈ P \setminus q ⊃ p ⇔ (q \vdash A \setminus q \vdash B)),$$

$$p \vdash ∀xA ≡ ∀xp \vdash A,$$

$$p \vdash ∃xA ≡ p \vdash ∃y A.$$

Thus, there is a correspondence between each expression A in the system $ZF^{(σ)}$ and the expression $p \vdash A$ in the system $ZF$.

In what follows we shall assume that the letters p, q, r, possibly with subscripts, denote variables running through P. For a given definition of constraint we can establish

$$ZF \vdash (p \vdash \neg \neg A ↔ p \vdash A)$$

and

$$ZF \vdash (p \vdash ∃xA ↔ ∀q (q ⊃ p3xφr ⊃ qr \vdash A)).$$

Let $U^n a(xφy) ≡ Ayφy(A(xφy) \setminus A(xφy) → y_1 = y_2)$.

**Lemma 1.**

$$ZF \vdash ∀zVp (p \vdash ∀xU^n a(xφy) → ∃q ⊃ p∀xxφy(q \vdash A(xφy) \setminus q \vdash \neg \neg A(xφy))).$$

**Proof (in ZF).** Let $p_φ \vdash ∀xU^n A(xφy)$ and let z be arbitrary. We define the functions $α(xφp)$ and $F(xφp)$:

$$α (xp) = \min \{α \mid q ⊃ p (rang(q) = α \setminus ∃y (q \vdash A(xφy) \setminus q \vdash \neg \neg A(xφy)))\},$$

$$F (xp) = \{q \mid q ⊃ p \setminus ∃y (q \vdash A(xφy) \setminus q \vdash \neg \neg A(xφy)) \setminus rang(q) = α (xp)\}.$$

If $p \vdash U^n a(xφy)$, then $F(xφp)$ is not empty. Indeed, there is a $y_0$ and a $q ⊃ p$ such that

$$q \vdash \neg \neg A(xφy_0) \setminus q \vdash \neg \neg A(y_0).$$

Suppose y is arbitrary and it is false that $q \vdash \neg \neg A(xφy)$. Then, by (1), $q \vdash A(xφy_0)$ and there is an $r ⊃ q r \vdash A(x, y)$. Since $r \vdash U^n A(xφy)$ and $r \vdash A(xφy)$, we have $r \vdash (y = y_0)$ and $y_0 = y$. Consequently, $q \vdash A (x, y)$. Thus, $α (xp) ∈ On$ and $F (xp) ≠ φ$.

Let f be a function mapping the set z in a one-to-one manner onto some ordinal $α_0$. We define the function $π, D(π) = α_0$:

$$π1) \; π (0) = \{p_φ\}.$$

$$π2) \; α (a + 1) = ∪ \{F (f (α), p) \mid p ∈ (α)\}, \; a + 1 < α_0.$$

$$π3) \; Let λ be limiting and h a function defined on λ such that

$$∀β < λ (h (β) ≡ β) \land ∀β' < λ (β' > β → h (β') ≡ h (β'))).$$

Then $∪ \{h (β) \mid β < λ\} ≡ (λ)$.

We have $∀α < α_0 (α + φ).$ If the function $h$ satisfies the premise of $π3$, then $∪ \{h (β) \mid β < λ\} = p_φ ∈ P$ and

$$∀x (< λ (x = f (γ)) → ∃y (p_α \vdash A(xφy) \setminus p_β \vdash \neg \neg A(xφy))).$$

The lemma will be proved if we take $h$ such that $D(h) = α_0$. 

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