In this paper we find the conditions such that from the form of the defining relations we can form a view of the structural definability of the semigroup specified by these defining relations.

The fundamental results of this paper are Theorems 2-4 which give some sufficient conditions for the structural definability of semigroups specified by a finite number of defining relations. These theorems are formulated in § 2. Their proofs are based on Theorem 1 of § 1 which is of independent interest.

§ 1. Structural Isomorphisms of Semigroups Which Can Be Expanded in a Matrix Bundle. It is said (cf., e.g., [1], p. 422) that the semigroup \( \Gamma \) is a matrix bundle of semigroups \( \Gamma_{\alpha \beta} \) \((\alpha \in I, \beta \in J)\), if \( \Gamma \) is the union of nonintersecting subsemigroups \( \Gamma_{\alpha \beta} \) and \( \Gamma_{\alpha_1 \beta_1} \cap \Gamma_{\alpha_2 \beta_2} = \Gamma_{\alpha_1 \beta_2} \) \((\alpha_1, \alpha_2 \in I; \beta_1, \beta_2 \in J)\). We shall say that the matrix bundle \( \Gamma \) of semigroups \( \Gamma_{\alpha \beta} \) \((\alpha \in I, \beta \in J)\) is a nonsingular matrix bundle if \( |I| > 1, |J| > 1 \). We denote the operation of generating subsemigroups in a semigroup by angle brackets.

THEOREM 1. Each structural isomorphism of the semigroup \( \Gamma \) with left or right cancellation which can be expanded in a nonsingular matrix bundle of semigroups \( \Gamma_{\alpha \beta} \) \((\alpha \in I, \beta \in J)\) is induced by one of its isomorphisms or antiisomorphisms.

Proof. For the sake of definiteness we shall assume that \( \Gamma \) is a semigroup with right cancellation. At once we note that \( \Gamma \) does not contain idempotent elements. Indeed, if \( e \) is an idempotent element of \( \Gamma \), for each \( x \in \Gamma \), by right cancellation, we have \( xe = x \), and thus, \( \{J\} = 1 \), which is impossible. Since \( \Gamma \) has no idempotent elements, a structural isomorphism \( \phi \), as is well known (cf., e.g., [2], p. 673) is induced by a one-to-one mapping \( \phi \) of the semigroup \( \Gamma \) on to \( \Gamma' \). We show that in this case \( \phi \) is an isomorphism or an antiisomorphism. The proof is in several stages.

1. If \( x \in \Gamma_{\alpha_1 \beta_1}, y \in \Gamma_{\alpha_2 \beta_2}, \) and \( \beta_1 \neq \beta_2, \) then
   \[ \phi(xy) \subseteq \{\phi(x) \phi(y), \phi(y) \phi(x)\}. \]

Suppose the element \( xy \) can be written in the alphabet \( \{x, y\} \) as a word \( f(x, y) \), distinct from the word \( xy \). It is easy to see that the word \( f(x, y) \) has the form \( g(x, y)y \), where \( g(x, y) \) is a word in the alphabet \( \{x, y\} \) distinct from \( x \). If \( g(x, y) \) is the empty word, then \( xy = y \), and thus, \( x^2 = xy \). By right cancellation we obtain \( x^2 = x \) which is a contradiction. Hence, \( g(x, y) \) is not the empty word, distinct from \( x \). Since \( xy = g(x, y)y \), we have \( x = g(x, y) \). It follows from this that \( g(x, y) \) has the form \( h(x, y)x \), where \( h(x, y) \) is not the empty word in the alphabet \( \{x, y\} \). Thus, \( x = h(x, y) \) and \( h(x, y)x = h^2(x, y)x \), i.e., \( h(x, y) = h^2(x, y) \), which is impossible. Thus, \( xy \) can be represented in the alphabet \( \{x, y\} \) by a word distinct from the word \( xy \). By Lemma 1 of [2], \( xy \) is the S-product of the elements \( x \) and \( y \) (cf. [2], p. 671); thus, by Corollary 1 of [2], and Lemma 2 of [2], the conclusion follows.

2. If \( x \in \Gamma_{\alpha_1 \beta_1}, y \in \Gamma_{\alpha_2 \beta_2}, \) and \( \alpha_1 \neq \alpha_2, \) then
   \[ \phi(xy) \subseteq \{\phi(x) \phi(y), \phi(y) \phi(x)\}. \]
Let $z$ be an arbitrary element of $\Gamma_{e_2 Z_2}$, where $Z_2 = \beta$.

By point 1, we have $\varphi(x \cdot yz) \in \{\varphi(x)(yz), \varphi(yz), \varphi(x)\}$ and $\varphi(yz) \in \{\varphi(y) \varphi(z), \varphi(z) \varphi(y)\}$. Assume for the sake of definiteness that $\varphi(xyz) = \varphi(x) \varphi(yz)$ (the considerations otherwise are completely analogous). Consider the possible subcases.

2.1. $\varphi(yz) = \varphi(y) \varphi(z)$.

Here $\varphi(xyz) = \varphi(x) \varphi(y) \varphi(z)$. Obviously, $\varphi(\Gamma_{e_1 Z_1} \cup \varphi(\Gamma_{e_2 Z_2})$ is a subsemigroup of the semigroup $\Gamma'$. Thus, $u = \varphi^{-1}(\varphi(x) \varphi(y)) \in \Gamma_{e_1 Z_1} \cup \Gamma_{e_2 Z_2}$. Since $uz \neq zu$, by point 1, we have $\varphi(xz) \varphi(zx) \in \{\varphi(uz), \varphi(zu)\}$. Then, $\varphi(xyz) = \varphi(x) \varphi(y) \varphi(z) = \varphi(u) \varphi(z) \in \{\varphi(uz), \varphi(zu)\}$, i.e., $xyz \in \{uz, zu\}$. It follows from this that $xyz = uz$, i.e., $xy = u$. Thus, $\varphi(xy) = \varphi(u) = \varphi(x) \varphi(y)$.

2.2. $\varphi(yz) = \varphi(x) \varphi(y) \varphi(z)$.

Here $\varphi(xyz) = \varphi(x) \varphi(y) \varphi(z)$. Since $xz \neq zx$, by point 1, we have $\varphi(xz) \varphi(zx) \in \{\varphi(xz), \varphi(zx)\}$, i.e., $xyz \in \{xzy, yzx\}$, which is impossible.

3. If $x_1, x_2 \in \Gamma_{e_1 Z_1}, y \in \Gamma_{e_2 Z_2}$ and $\alpha_1 \neq \alpha_2, \beta_1 \neq \beta_2$, then $\varphi(yx_1) = \varphi(y) \varphi(x_1)$ implies that $\varphi(yx_2) = \varphi(y) \varphi(x_2)$.

Let $\varphi(yx_1) = \varphi(y) \varphi(x_1)$ and $\varphi(yx_2) = \varphi(x_2) \varphi(y)$. It is easy to see that $\varphi(x_2y) = \varphi(y) \varphi(x_2)$. Noting the results of points 1 and 2, we obtain the following diagram of possible equations for the element $\varphi(yx_1x_2y)$ (the arrows denote equations):

The inclusion $yx_1x_2y \in \{yx_1x_2, yx_2x_1, y^2x_1x_2, x_1x_2y^2, x_1x_2y^2, yx_1yx_1, yx_2yx_2\}$ follows from the diagram, but is impossible.

4. Let $x \in \Gamma_{e_1 Z_1}, z \in \Gamma_{e_2 Z_2}$, $y \in \Gamma_{e_2 Z_2}$ and $\alpha_1 \neq \alpha_2, \beta_1 \neq \beta_2$. Then $\varphi(xy) = \varphi(x) \varphi(y)$ implies that $\varphi(xz) = \varphi(x) \varphi(z)$.

Let $\varphi(xy) = \varphi(x) \varphi(y)$ and $\varphi(xz) = \varphi(x) \varphi(z)$. By point 2, we have $\varphi(xz) \varphi(xz) \in \{\varphi(xy), \varphi(xz), \varphi(xz) \varphi(y)\}$. If $\varphi(xyz) = \varphi(xz) \varphi(y)$, then $\varphi(xyz) = \varphi(xz) \varphi(y) = \varphi(z) \varphi(x) \varphi(y) = \varphi(z) \varphi(x) \varphi(y) \in \{\varphi(xz), \varphi(x), y\}$, from which $yxz \in \{z, xz\}$, which is impossible. Consequently, $\varphi(xyz) = \varphi(x) \varphi(xz) = \varphi(y) \varphi(z) \varphi(x)$. Noting points 1 and 2, we obtain the following diagram of possible equations for $\varphi(x) \varphi(y) \varphi(z)$ (again the arrows denote equations):