\[ R(\lambda) = \frac{A_{-1}}{(\lambda - \lambda_0)^k} + \ldots + \frac{A_{-r}}{(\lambda - \lambda_0)^m} + \Phi(\lambda), \]

where \( \Phi(\lambda) \) is an analytic function inside \( \Gamma \), and \( A_{-r} = -(1/(2\pi i)) \int_\Gamma (\lambda - \lambda_0)^{-r} R(\lambda) d\lambda \), then from Lemma 3, with \( f = 1 \) and \( g = (\lambda - \lambda_0)^{-r-1} \), we obtain

\[ P(1/(2\pi i)) \int_\Gamma (\lambda - \lambda_0)^{-r} B R(\lambda) d\lambda = (1/(2\pi i)) \int_\Gamma (\lambda - \lambda_0)^{-r} R(\lambda) d\lambda = -A_{-r}. \]

From Lemma 5 with \( f = 1 \), we find that

\[ [(A - \lambda_0 B) P]^{r-1} = (1/(2\pi i)) \int_\Gamma (\lambda - \lambda_0)^{-r} B R(\lambda) d\lambda. \]

Consequently, \( P[(A - \lambda_0 B) P]^{r-1} = -A_{-r} \), and since

\[ A_{-r} \neq 0, \text{ then } [P(A - \lambda_0 B)]^{r-1} P \neq 0. \]

Now let \( M_1 \) be a finite-dimensional subspace of the space \( E \). We show that, in this case, the expansion of \( R(\lambda) \) contains only a finite number of terms with negative powers of \( \lambda - \lambda_0 \). In fact, consider \( P(A - \lambda B) \) on elements of \( M_1 \). Since the operator \( P_1 \) is the identity operator in this subspace, then, putting \( PA = \bar{A} \), we get \( P(A - \lambda B) = \bar{A} - \lambda \bar{I} \) in \( M_1 \). Inside \( \Gamma \) lies the unique singular point of the resolvent, and hence, applying the theorem, we find that the spectrum of \( (A - \lambda B)M \) consists of the single point \( \lambda_0 \). Since, in the finite-dimensional case, \( \lambda_0 \) is an eigenvalue of \( (A - \lambda B)M \), i.e., \( (A - \lambda_0 B)M x_0 = 0 \), then \( P(A - \lambda_0 B)M x_0 = (\bar{A} - \lambda_0 \bar{I}) x_0 = 0 \), and the single point \( \lambda_0 \) makes up the spectrum of \( \bar{A} - \lambda \bar{I} \), considered in \( M_1 \). We know (see [1]) that, in this case, \( (\bar{A} - \lambda_0 \bar{I})^n = 0 \) for some positive integer \( n \), i.e., \( [P(A - \lambda_0 B)]^{r-1} P = 0 \). Hence, if \( \lambda_0 \) is a fixed unique singular point of the resolvent \( R(\lambda) \) lying inside \( \Gamma \), the operator \( P(A - \lambda_0 B) \), considered in \( M_1 \), is nilpotent.

LITERATURE CITED


MONOTONE TRANSFORMATIONS AND DIFFERENTIAL PROPERTIES OF FUNCTIONS

L. I. Kaplan and S. G. Slobodnik

The class of all real functions of a single variable which become everywhere differentiable after a certain homeomorphic transformation of coordinate axis is described. Moreover, various examples about differential properties of functions are given (in particular, an elementary construction of a nonconstant continuously differentiable real function of two variables, every value of which is critical — the example of Whitney, is given).

We will consider real functions defined on segments of line \( \mathbb{R}^1 \) and by differentiability we will mean the existence of a finite derivative.

If \( f \) is defined on a segment \( \Delta = [a, b] \) \((a < b)\) and \( E \subset \Delta \), then we will denote by \( \omega(f; E) \) the oscillation of the function \( f \) on \( E \) and by \( V(f; E) \) the total variation of \( f \) on \( E \). If \( V(f; A) < +\infty \), then we set \( v_f(x) = V(f; [a, x]) \) and call \( v_f \) the total variation function of \( f \).

By \( H[\alpha, \beta] \) we will denote the set of functions which are defined, strictly monotone, and differentiable on \([\alpha, \beta] \). Let us set \( H = \bigcup_{c < d} H[c, d] \).

Principal Information Computation Center of the State Standard Office of the USSR.
LEMMA 1 (Luzin—Men’shov). Let a closed set \( F \subseteq G \subseteq \mathbb{R}^1 \), where \( G \) is measurable and the density of the set \( G \) is 1 at each point \( x \in F \). Then there exists a closed set \( E, F \subseteq E \subseteq G \), such that the density of the set \( E \) is 1 at each point \( x \in F \) and the density of the set \( G \) is also 1 at each point \( x \in E \).

Using this lemma, Bogomolova [1] has proved the following lemma.*

LEMMA 2. If \( K_1 \) and \( K_2, K_1 \subseteq K_2 \), are closed subsets of a segment \([a, b]\) and the density of the set \( K_2 \) is equal to 1 at each point \( x \in K_1 \), then there exists an asymptotically continuous function \( \Omega \) on \([a, b]\) such that \( \Omega(x) = 0 \) for \( x \in F \) and \( \Omega(x) = 1 \) for \( x \in M \).

The following corollary follows from Lemmas 1 and 2.

COROLLARY 1. Let \( M \) be a subset of \([a, b]\) of measure zero. Then for an arbitrary closed set \( F \subseteq [a, b] \setminus M \) there exists an asymptotically, continuous function \( \Omega \), \( 0 \leq \Omega \leq 1 \), defined on \([a, b]\) such that \( \Omega(x) = 0 \) for \( x \in F \) and \( \Omega(x) = 1 \) for \( x \in M \).

Proof. Since the set \( CM = \mathbb{R}^1 \setminus M \) consists of the characteristic density points, it follows from Lemma 1 that there exists a closed set \( E, F \subseteq E \subseteq CM \), such that the density of \( E \) is equal to 1 at every point \( x \in F \). Now, applying Lemma 2 to the set \( F \subseteq E \), we prove the existence of the desired function.

LEMMA 3. For an arbitrary set \( M \subseteq [a, b] \) of measure zero there exists a function \( f \in H[a, b] \) such that \( f'(x) = 0 \) for \( x \in M \).

Proof. Let \( \{F_i\} \) be a sequence of closed subsets of \([a, b]\), each of which is disjoint from \( M \), such that \( \text{mes } F_i + (b - a) \rightarrow +\infty \). By virtue of Corollary 1, for every \( i \) there exists an asymptotically continuous function \( \Omega_i \), \( 0 \leq \Omega_i \leq 1 \), which is equal to 1 on \( M \) and 0 on \( F_i \). We set

\[
\Omega = (1 - \Omega_1) + \frac{1}{2} (1 - \Omega_2) + \frac{1}{4} (1 - \Omega_4) + \ldots
\]

The function \( \Omega \) is asymptotically continuous and bounded as the limit of a uniformly convergent sequence of asymptotically continuous and bounded functions, vanishes on \( M \), and is positive on \( \bigcup F_i \), i.e., almost everywhere on \([a, b]\). Consequently, the function \( f(x) \), defined on \([a, b]\) by the equation

\[
f(x) = \int_a^x \Omega(t) \, dt,
\]

satisfies the requirements of the lemma.

LEMMA 4 (Zagorskii). Let \( M \) be a \( G_\delta \)-subset of \([a, b]\) of measure zero. Then there exists an increasing continuous function \( g(x) \) which is defined in \([a, b]\) and has a finite derivative for \( x \in [a, b] \setminus M \) and an infinite derivative for \( x \in M \).

In view of the importance of this lemma in the sequel, we give its proof.

Proof. It is easily shown that \( M \) can be represented in the form \( M = \bigcap_{n=1}^{\infty} G_n \), where each \( G_n \) is open in \([a, b]\), \( G_{n+1} \subseteq G_n \) \((n = 1, 2, \ldots)\) and \( \text{mes } (G_{n+1} \setminus l) < \text{mes } l/2^n \) for every segment \( l \) such that \( l \cap CG_n \neq \emptyset \).

Using Corollary 1, for each \( n \) we choose an asymptotically continuous function \( \Omega = \sum_{n=1}^{\infty} \Omega_n \) with \( \Omega_n = 1 \), if \( x \in M \), and \( \Omega_n(x) = 0 \) outside \( G_n \). We set \( \Omega = \sum_{n=1}^{\infty} \Omega_n \).

Without loss of generality we can assume that \( G_1 \neq [a, b] \). Therefore, on taking the segment \([a, b]\) as \( l \), it follows from (1) that \( G_{n+1} \leq (b - a)/2^n \). Since \( \Omega_n \leq \chi_n \), where \( \chi_n \) is the characteristic function of the set \( G_n \), it follows that

\[
\sum_{n=1}^{\infty} \Omega_n(t) dt \leq \sum_{n=1}^{\infty} \text{mes } G_n < +\infty.
\]

*See also the proof of Lemma 1 in the second part of [2].