In this paper we give the proof of the theorem, formulated in [1], on the local $C^\omega$-solvability of the system of partial differential equations

$$(V\psi)(x) = g(x, \psi(x)), \quad (1)$$

where $V$ is a local vector field of class $C^\omega$, defined in the neighborhood of its singular point $x = 0$ in $\mathbb{R}^n$, while $g: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ is a given $C^\omega$-mapping.

A necessary condition for the solvability of Eq. (1) is its formal solvability, i.e., the solvability in formal Taylor series at the point $x = 0$. Assume that Eq. (1) is formally solvable and let $\psi_0$ be a formal solution of it. We consider some $C^\omega$-mapping $\psi: \mathbb{R}^n \to \mathbb{R}^m$ with Taylor series $\psi_0$ at the origin. The change $\varphi \to \psi + \psi_0$ of the unknown mapping reduces Eq. (1) to the equation

$$(V\psi)(x) = \tilde{g}(x, \psi(x)). \quad (2)$$

where $\tilde{g}(x, y) = g(x, y + \psi_0(x)) - (V\psi_0)(x)$. Moreover, the mapping $\tilde{g}(x, 0)$ has zero Taylor series at the origin. If Eq. (2) has a $C^\omega$-solution $\psi$, then the initial equation has the $C^\omega$-solution $\psi + \psi_0$. If, in addition, $\dot{\varphi} = 0$, then the Taylor series at zero of the solution $\psi + \psi_0$ is equal to the formal solution $\tilde{\psi}_0$. In this case we shall say that the formal solution of Eq. (1) is restored to a local $C^\omega$-solution of this equation.

A diffeomorphism $F: \mathbb{R}^n \to \mathbb{R}^n$ is said to be quasihyperbolic of order $k \geq 0$, if there exists a decomposition $\mathbb{R}^n = L_- + L_+$ of the space $\mathbb{R}^n$ into a direct sum of subspaces, invariant with respect to $F$, such that in some norm we should have

1. $\rho(F(x), Z_-) \leq \rho(x, Z_-)(1 - c_1 \|x\|^k),$
2. $\rho(F^{-1}(x), Z_-) \leq \rho(x, Z_-)(1 - c_2 \|x\|^k),$
3. $\|F'(x)\| \leq 1 + a \|x\|^k, \quad \|F^{-1}(x)\| \leq 1 + b \|x\|^k.$

Here $c_1, c_2, a, b$ are some positive numbers, while $\rho$ is the distance from a point to the corresponding subspace. For $k = 0$ the diffeomorphism is said to be hyperbolic. If $L_- = 0$ (resp., $L_+ = 0$), then $F$ is called a quasicontraction (resp., quasiextension).

A field $V$ is said to be quasihyperbolic of order $k$ if its flow $F^t$ is quasihyperbolic of order $k$ for each $t \neq 0$. 

Example 1. The field \( V(x) = \|x\|^2, x \in \mathbb{R}^n \), is quasihyperbolic. Its flow has the form
\[
F_t(x) = x(1 + t\|x\|^2 + \ldots).
\]
Therefore, \( F_t \) is a quasicontraction of order 2 for \( t < 0 \) and a quasi-extension of order 2 for \( t > 0 \).

Example 2. The field \( V(x) = (-\|x\|^2\xi, \|x\|^2\eta), x = (\xi, \eta) \in \mathbb{R}^2, k \in \mathbb{Z}^+ \), is quasihyperbolic of order \( 2k \), since its flow has the form
\[
F_t(x) = (\xi(1 - t\|x\|^{2k} + \ldots), \eta(1 + t\|x\|^{2k} + \ldots)).
\]

Theorem 1. Assume that in Eq. (1) the field \( V \) is quasihyperbolic of order \( k \), while

\[Q(x) = O(||x||^k)\]

satisfies the estimate \( Q(x) = O(||x||^k) \). Then the formal solution \( \bar{\varphi}_0 \) is restored to a local \( C^\infty \)-solution of Eq. (1).

Proof. We recall that a mapping \( f \) is said to be planar on the set \( M \) if \( f^{(s)}(x) = 0, x \in M, s = 0, 1, 2, \ldots \). In order to prove Theorem 1 it is sufficient to prove the existence of a solution, planar at zero, of Eq. (1) under the conditions \( g(x, 0) = 0 \) and \( Q(x) = O(||x||^k) \). We set \( g(x, 0) = g_+(x) + g_-(x) \), where \( g_+ \) is a mapping, planar on the subspace \( L_+ \), respectively. We consider the equation
\[
(V\varphi)(x) = g_+(x, \varphi(x))
\]
where \( g_+(x, y) = g(x, y) - g_-(x, y) \). Here the mapping \( g_+(x, 0) = g_+(x) \) is planar on the subspace \( L_+ \). Let \( \varphi_0 \) be a solution of Eq. (3). We consider the equation
\[
(V\varphi)(x) = g_-(x, \varphi(x))
\]
where \( g_-(x, y) = g(x, y + \varphi(x)) - g(x, \varphi(x)) + g_-(x) \). Here the mapping \( g_-(x, 0) = g_-(x) \) is planar on \( L_+ \). If \( \varphi_- \) is a solution of Eq. (4), then \( \varphi = \varphi_+ + \varphi_- \) is a solution of the initial equation. Thus, in order to prove Theorem 1 it is sufficient to prove the solvability of Eqs. (3), (4).

First we consider Eq. (3). We set \( V = \sum_{i=1}^n \varphi_i(x) \frac{\partial}{\partial x_i}, g_+(x, y) = (g_+^1(x, y), \ldots, g_+^m(x, y)) \). We consider the vector field \( W = (\sum_{i=1}^n \varphi_i(x) \frac{\partial}{\partial x_i}, \sum_{i=1}^m g_+^i(x, y) \frac{\partial}{\partial y_j}) \) in \( \mathbb{R}^n \times \mathbb{R}^m \). Let \( H^t(x, y) = (F^t(x), G^t(x, y)) \) be its flow. If \( \varphi \) is a solution of equation
\[
F^t(x) = G^t(x, \varphi(x))
\]
for all sufficiently small \( t \), then \( \varphi \) is a solution of Eq. (3). Let \( V = (V_+, V_-) \) be the coordinate notation of the field \( V \) according to the decomposition \( \mathbb{R}^n = L_+ + L_- \). We consider the \( C^\infty \)-functions \( \tau_+(x) \) and \( \tau_-(x, y) \), equal to unity in some neighborhood of the origin and equal to zero outside some large neighborhood. It is sufficient to prove the solvability of the equation obtained from (3) by the substitution
\[
V(x) \to (V_+(x), \tau_+(x) V_-(x)), g_+(x, y) \to \tau_+(x, y) g_+(x, y).
\]
In this case the flow \( H^t(x, y) \) will possess the following properties.

1. There exists a neighborhood \( U \subset \mathbb{R}^n \) of the origin, invariant with respect to \( F^t; \)
2. \( p(F^t(x), L_-) \leq (1 - c(t)||x||^k)p(x, L_-); \)
3. \( G^t(x, y) = y, x \in U. \)

Setting \( t = 1 \) in (5), we prove the \( C^\infty \)-solvability of the obtained equation in the neighborhood \( U \). To this end we rewrite Eq. (5) for \( t = 1 \) in the form
\[
\varphi(x) = q^1(x) \varphi(F(x)) + h(x, \varphi(x)) + \gamma(x),
\]
separating the linear and the quadratic parts with respect to \( y \) \( (h(x, y) = O(||y||^2)) \). We note that the free term \( \gamma(x) \) in (6) is a planar mapping on the subspace \( L_- \). Therefore, the operator \( A, \) in the right-hand side of Eq. (6), acts in the space of \( C^\infty \)-mappings \( \varphi: U \to \mathbb{R}^m \), planar on the intersection \( U \cap L_- \). We fix a nondecreasing sequence of nonnegative integers \( \{v_\nu\} \) and an infinite matrix \( \{C_{\nu \nu}\} \). The set \( K = K(\{C_{\nu \nu}\}, \{v_\nu\}) \) of all \( C^\infty \)-mappings \( \varphi: U \to \mathbb{R}^m \), for which
\[
\|\varphi\|_{V_\nu} = \max_{x \in L_-, \nu < v} \|\varphi^{(\nu)}(x)\|_{L_-} \leq C_{\nu \nu} \nu \geq v_\nu,
\]
*Here \( \bar{\varphi}_0 \) is the Borel extension of the formal solution \( \varphi_0 \).