A polyhedral function \( l \Phi(\Delta_n) \) (\( f \)), interpolating a function \( f \), defined on a polygon \( \Phi \), is defined by a set of interpolating nodes \( \Delta_n \subset \Phi \) and a partition \( P(\Delta_n) \) of the polygon \( \Phi \) into triangles with vertices at the points of \( \Delta_n \). In this article we will compute for convex moduli of continuity the quantities

\[
E(H^0_\Phi; P(\Delta_n)) = \sup_{f \in H^0_\Phi} \| f - l \Phi(\Delta_n) \|, \tag{1}
\]

and also give an asymptotic estimate of the quantities

\[
E_n(H^0_\Phi) = \inf_{\Delta_n} \inf_{P(\Delta_n)} E(H^0_\Phi; P(\Delta_n)). \tag{2}
\]

1. Let \( \Phi \) be a polygon in the plane \( \mathbb{R}^2 \), \( N_\Phi \) be the number of vertices of this polygon; \( \Delta_n \) (\( n = N_\Phi, N_\Phi + 1, \ldots \)) be the set \( \{M_1, \ldots, M_n\} \) of points of the polygon \( \Phi \) containing all its vertices. To each partition \( P(\Delta_n) \) of the polygon \( \Phi \) into triangles, no two of which have any common interior point and which have vertices at the points of the set \( \Delta_n \), and to every function \( f \) defined on \( \Phi \) we associate a function \( l \Phi(\Delta_n) \) (\( f \)) as follows: If a triangle \( T \) with vertices \( A_1 = (x_1, y_1), A_2 = (x_2, y_2), \) and \( A_3 = (x_3, y_3) \) belongs to the partition \( P(\Delta_n) \), then the function \( l \Phi(\Delta_n) \) (\( f \)) describes on this triangle the plane passing through the points \( (x_1, y_1, f(x_1, y_1)), (x_2, y_2, f(x_2, y_2)), \) and \( (x_3, y_3, f(x_3, y_3)) \).

If \( M_1 = (x_1, y_1) \) and \( M_2 = (x_2, y_2) \) are points of the plane, then

\[
\rho(M_1, M_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.
\]

By \( H^0_\Phi \) we will denote the class of functions \( f \) defined on \( \Phi \) such that

\[
\forall M_1, M_2 \in \Phi \quad |f(M_1) - f(M_2)| \leq \omega(\rho(M_1, M_2)),
\]

where \( \omega(t) \) is a given modulus of continuity. If \( A \subset \Phi \), then

\[
H^0_\Phi(A) = \{ f \in H^0_\Phi : \forall M \in A \quad f(M) = 0 \}. \tag{1}
\]

Let us set

\[
E(H^0_\Phi, P(\Delta_n)) = \sup_{f \in H^0_\Phi} \| f - l \Phi(\Delta_n)(f) \|, \tag{1}
\]

where \( \| \cdot \| \) is the Chebyshev norm, and

\[
E_n(H^0_\Phi) = \inf_{\Delta_n} \inf_{P(\Delta_n)} E(H^0_\Phi, P(\Delta_n)). \tag{2}
\]

In Sec. 2 we will compute the exact values of the quantities (1). In Secs. 3 and 4 we will compute the asymptotically exact values (as \( n \to \infty \)) of the quantities (2). In Sec. 5 we will deal with summation formulae.

2. Let $\Delta_n$ be a set of points of a polygon $\Phi$ containing all its vertices and $P(\Delta_n)$ be the partition of the polygon $\Phi$ into triangles. Let us consider the set of circumcircles of the acute-angled triangles of the partition $P(\Delta_n)$. Let $r(P(\Delta_n))$ denote the greatest of the radii of these circles. For every right-angled or obtuse-angled triangle of the partition $P(\Delta_n)$ we find the length of the greatest side and denote the greatest of these lengths by $l(P(\Delta_n))$. Let us set $h(P(\Delta_n)) = \max \{r(P(\Delta_n)), 2^{-1}l(P(\Delta_n))\}$. The following theorem holds.

**THEOREM 1.** If $\omega(t)$ is a modulus of continuity which is convex upwards, then

$$E(\mathbb{H}_n, P(\Delta_n)) = \omega(h(P(\Delta_n))), \quad n = N_0, N_0 + 1, \ldots$$

**Proof.** Let us consider an arbitrary triangle of the partition $P(\Delta_n)$. Let $A_1 = (x_1, y_1)$, $A_2 = (x_2, y_2)$ and $A_3 = (x_3, y_3)$ be the vertices of this triangle. Every point $M = (x, y)$ of this triangle is a convex combination of its vertices

$$M = \lambda_1 A_1 + \lambda_2 A_2 + \lambda_3 A_3 = (\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3, \lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3),$$

where $\lambda_1 + \lambda_2 + \lambda_3 = 1$ and $\lambda_1, \lambda_2, \lambda_3 \geq 0$. Therefore $\forall f \in H^\infty$, we have

$$|f(M) - l(P(\Delta_n)) f(M)| \leq \lambda_1 |f(M) - f(A_1)| + \lambda_2 |f(M) - f(A_2)| + \lambda_3 |f(M) - f(A_3)| \leq \lambda_1 \omega(V(x - x_1)^2 + (y - y_1)^2) + \lambda_2 \omega(V(x - x_2)^2 + (y - y_2)^2) + \lambda_3 \omega(V(x - x_3)^2 + (y - y_3)^2).$$

Since $\omega(t)$ and $\sqrt{t}$ are functions which are convex upwards, it follows that for every point $M$ of the triangle under consideration we have

$$|f(M) - l(P(\Delta_n)) f(M)| \leq \omega(V(x - x_1)^2 + (y - y_1)^2) + \lambda_2 \omega(V(x - x_2)^2 + (y - y_2)^2) + \lambda_3 \omega(V(x - x_3)^2 + (y - y_3)^2)).$$

Without loss of generality, we may assume that $(x_1, y_1) = (0, 0)$, $(x_2, y_2) = (c, 0)$, and $(x_3, y_3) = (a, b)$. Then the expression under the radical sign in (4) can be written as a function of only $\lambda_1, \lambda_2$, and $\lambda_3$ in the following form:

$$G(\lambda_1, \lambda_2, \lambda_3) = \lambda_1 [(c x_1 + a x_2 + b)^2] + \lambda_2 [(c_2 + a x_2 - c)^2 + b^2] + \lambda_3 [(c_2 + a x_3 - a)^2 + b^2 (x_3 - 1)^2].$$

It is easily seen that the only maximum of this function under the condition $\lambda_1 + \lambda_2 + \lambda_3 = 1$ is attained for

$$\lambda_1 = \frac{1}{2} - \left(1 - \frac{a}{c}\right) \frac{a^2 + b^2 - ac}{2b^2}, \quad \lambda_2 = \frac{1}{2} - \frac{a^2 + b^2 - ac}{2b^2}, \quad \lambda_3 = \frac{a^2 + b^2 - ac}{2b^2}.$$

These numbers, when substituted in Eq. (3), determine the point $M_0$ which is the center of the circumcircle of the triangle under consideration. If this triangle is acute-angled, then the maximum of the function $G(\lambda_1, \lambda_2, \lambda_3)$ under the conditions $\lambda_1 + \lambda_2 + \lambda_3 = 1$ and $\lambda_1, \lambda_2, \lambda_3 \geq 0$ is attained for the same $\lambda_1, \lambda_2, \lambda_3$ as for the maximum of the function $G(\lambda_1, \lambda_2, \lambda_3)$ under the condition $\lambda_1 + \lambda_2 + \lambda_3 = 1$. In this case, from (4) for each of its points we get

$$|f(M) - l(P(\Delta_n)) f(M)| \leq \omega(R),$$

where $R$ is the radius of the above circumcircle. But if the triangle under consideration is obtuse angled or right angled, then the maximum of the function $G(\lambda_1, \lambda_2, \lambda_3)$ under the conditions $\lambda_1 + \lambda_2 + \lambda_3 = 1$ is attained when at least one of $\lambda_1, \lambda_2, \lambda_3$ is zero. In this case, as is easily computed,

$$\max G(\lambda_1, \lambda_2, \lambda_3) \leq (2^{-1})^3,$$

where the maximum is taken over all points $M$ of the triangle under consideration and $l$ is the length of its greatest side. Since we have considered an arbitrary triangle, it follows from (5) and (6) that

$$E(\mathbb{H}_n, P(\Delta_n)) \leq \omega(h(P(\Delta_n))).$$

For the completion of the proof of the theorem it remains to obtain a lower bound for the quantity $E(\mathbb{H}_0, P(\Delta_n))$. We determine the point $M_0$ in the following manner. If $h(P(\Delta_n)) = r(P(\Delta_n))$, then we consider an acute-angled triangle of the partition $P(\Delta_n)$, the radius of whose circumcircle is equal to $r(P(\Delta_n))$, and we choose the center of this triangle as $M_0$. But if $h(P(\Delta_n)) = 2^{-1}l(P(\Delta_n))$, then we consider a right-angled or an obtuse-angled triangle of the partition $P(\Delta_n)$, the length of whose greatest side is equal to $l(P(\Delta_n))$, and we take the middle point of this side as $M_0$. Let us set

$$F_0(M) = \begin{cases} \omega(h(P(\Delta_n)) - r(M, M_0)), & r(M, M_0) \leq h(P(\Delta_n)), \\ 0, & r(M, M_0) > h(P(\Delta_n)). \end{cases}$$